Beginning Algebra Part II

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Open Textbook Collaborative

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The courses align to [career pathways in New Jersey's growth industries]() including health services, technology, energy, and global manufacturing and supply chain management as identified by the New Jersey Council of Community Colleges.

<u>Table of Contents</u>

6.1 Factoring-Greatest Common Factor

Learning Objectives: In this section, you will:

- Find the Greatest Common Factor of a list of numbers.
- Find the Greatest Common Factor of a list of variable terms.

Factoring the Greatest Common Factor of a Polynomial:

When we study fractions, we learn that the **greatest common factor** (GCF) of two numbers is the largest number that divides evenly into both numbers.

For instance, 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20. The CGF of polynomials works the same way: 4x is the GCF of $16x$ *and* $20x^{2.}$ Because it is the largest polynomial that divides evenly into both $16x$ *and* $20x^2$

When factoring a polynomial expression, our first step should be to check for a GCF. Look for the GCF of the coefficients, and then look for the GCF of the variables.

Find the GCF of a list of numbers.

Example A) Find the GCF for 60, 45.

$60 = 2\cdot2\cdot3\cdot5$
$45 = 3\cdot3\cdot5$
$GCF = 3\cdot5 = 15$

Example B) Find the GCF for 20, 13,15

$20=2\cdot2\cdot5$
$13=13$
$15=3\cdot5$
GCF=1 (there are no primes common to all three numbers, so the CGF is 1

Find the GCF of a list of variable terms.

The terms x^6, x^{11} have x^6 AS the GCF since the smaller power on the variable x in the factored forms is 6.

Example C) Find the GCF for $x^3y^3z^5$ *and* $x^2yz^{10.}$

GCF = x^2yz^5

Example D) Find the GCF for $60x^3y^3z^5$ *and* $45x^2yz^{10.}$

GCF= $15x^2yz^5$

Factoring by GCF

The opposite of multiplying polynomials together is factoring polynomials. There are many benefits of a polynomial being factored. To factor a number is to write it as a product of two or more numbers.
When we multiplied polynomials, we multiplied monomials by polynomials by distributing, solving problems such as $4x^2(2x^2 - 3x + 8) = 8x^2 - 13x^3 + 32$.

Example E) Write in factored form by GCF: $10y^2+35y^3$

since GCF=5y
$10y^2+35y^3 = 5y(2y) +5(7y^2) = 5y(2y+7y^2)$

Check: $5y(2y+7y^2)$ Distributive property, multiply the factored form.
$= 5y(2y) + 5(7y^2)$
$= 10y^2+35y^3$

Example F) $30m^5+20m^4+15m^3$
GCF=$5m^3$
$30m^5+20m^4+15m^3 = 5m^3(6m^2) + 5m^3 (4m) + 5m^3(3) = 5m^3(6m^2+ 4m+3)$

Example G) $-8x^3+24x^2+16x$
GCF = $-8x$
$-8x^3+36x^2+16x = -8x(x^2)-8x(4x)-8x (2)= -8x(x^2+4x-2)$

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Worksheet 6.1: Factoring-Greatest Common Factor

Factor the common factor out of each expression.

1) $9 + 8b^2$

2) $x - 5$

3) $45x^2 - 25$

4) $1 + 2n^2$

5) $56 - 35p$

6) $50x - 80y$

7) $7ab - 35a^2b$

8) $27x^2y^5 - 72x^3y^2$

9) $-3a^2b + 6a^3b^2$

10) $8x^3y^2 + 4x^3$

11) $-5x^2 - 5x^3 - 15x^4$

12) $-32n^9 + 32n^6 + 40n^5$

13) $20x^4 - 30x + 30$

14) $21p^6 + 30p^2 + 27$

15) $28m^4 + 40m^3 + 8$

16) $-10x^4 + 20x^2 + 12x$

6.2 Factoring-Grouping

Learning Objectives: In this section, you will:

- Factor polynomials with four terms using grouping.

Factoring by Groping

When a polynomial has 4-terms, the GCF might be used to factor by grouping.

(2a+3) (5b+2)	Distribute (2a+3) into second parenthesis.
5b(2a+3) +2 (2a+3)	Distribute each monomial.
10ab+15b+4a+6	Our solution

The solution has four terms in it. We arrived at the solution by looking at the two parts, 5b (2a + 3) and 2(2a + 3). When we are factoring by grouping, we will always divide the problem into two parts, the first two terms and the last two terms. Then we can factor the GCF out of both the left and right sides. When we do this our hope is what is left in the parenthesis will match on both the left and right. If they match, we can pull this matching GCF out front, putting the rest in parenthesis and we will be factored. The next example is the same problem worked backwards, factoring instead of multiplying.

Example A) Factor *10ab+15b+4a+6*

10ab+15b+4a+6	Split problem into two groups.
10ab+15b+4a+6	GCF on left is 5b, on the right is 2.
5b(2a+3) +2 (2a+3)	(2a+3) is the same! Factor out this GCF.
(2a+3) (5b+2)	Our Solution (factored form)

The key for grouping to work is after the GCF is factored out of the left and right, the two binomials must match exactly. If there is any difference between the two, we either have to do some adjusting or it can't be factored using the grouping method.

Example B) Factor *5xy-8x-10y+16*

5xy-8x -10y+16	Split problem into two groups.
5xy-8x -10y+16	GCF on left is *x*, on right we need a negative, so we use -2
x(5y-8) -2(5y-8)	(5y-8) is the same! Factor out this GCF.
(5y-8) (x-2)	Our Solution

Example C) Factor $6x^3-15x^2+2x-5$

$6x^3-15x^2 \quad +2x-5$	Split problem into two groups.
	GCF on left is $3x^2$ *on right, no GCF, use* 1.
$3x^2(2x-5)+1(2x-5)$	*(2x-5)* is the same and facotor out this GCF
$(2x-5)(3x^2+1)$	Our Solution

Worksheet 6.2: Factoring Polynomial by Grouping

Factor each completely.

1) $40r^3 - 8r^2 - 25r + 5$

2) $35x^3 - 10x^2 - 56x + 16$

3) $3n^3 - 2n^2 - 9n + 6$

4) $14v^3 + 10v^2 - 7v - 5$

5) $15b^3 + 21b^2 - 35b - 49$

6) $6x^3 - 48x^2 + 5x - 40$

7) $3x^3 + 15x^2 + 2x + 10$

8) $28p^3 + 21p^2 + 20p + 15$

9) $35x^3 - 28x^2 - 20x + 16$

10) $7n^3 + 21n^2 - 5n - 15$

6.3 Factoring-Trinomials where *a=1*

Learning Objectives: In this section, you will:

- Factor trinomials with coefficient a=1
- Factor trinomials after factoring out the GCF.

Factoring with three terms, or trinomials, is the most important type of factoring to be able to master. As factoring is multiplication backwards, we will start with a multiplication problem and look at how we can reverse the process.

$(x+6)(x-4)$	Distribute (x +6) through second parenthesis.
$x(x+6)-4(x+6)$	Distribute each monomial through parenthesis
$x^2+6x-4x-24$	Combine like terms.
$x^2+2x-24$	Our Solution

The trick to making these problems work is how we split the middle term. Why did we pick $+6x-4x$ and not $+5x-3x$? The reason is because 6x − 4x is the only combination that works! So how do we know what is the one combination that works? To find the correct way to split the middle term we will use what is called the ac method. In the next lesson we will discuss why it is called the ac method. The way the ac method works is we find a pair of numbers that multiply to a certain number and add to another number. Here we will try to multiply to get the last term and add to get the coefficient of the middle term. In the previous 1 example that would mean we wanted to multiply to − 24 and add to + 2. The only numbers that can do this are 6 and − 4 ($6\cdot-4=-24$ and $6+(-4)=2$).

Example A) Factor completely $x^2+9x+18$

$x^2+9x+18$	Want to multiply to 18, add to 9.
$x^2+6x+3x+18$	6 and 3, split the middle term.
$x(x+6)+3(x+6)$	Factoring by grouping
$(x-3)(x-1)$	

Example B) Factor completely $x^2-8x-20$

$x^2-8x-20$	Want to multiply to -20, add to -8
$x^2-10x+2x-20$	-10 and 2, split the middle term.
$x(9x-10)+2(x-10)$	Factor by Grouping
$(x-10)(x+2)$	Factored form

Example C) Factor completely $x^2+10x+21$

$x^2+10x+21$
$x^2+7x+3x+21$
$x(x+7)+3(x+7)$
$(x+3)(x+7)$

Example D) Factor completely $m^2-mn-30n^2$

$m^2-mn-30n^2$	Want to multiply to − 30, add to – 1; 5 and − 6, Write the factors, don′t forget the second variable.
$(m+5n)(m-6n)$	Our Solution

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Worksheet 6.3: Factoring-Trinomials where *a=1*

Factor each completely. (Remember to pull out the GCF first.)

1) $p^2 + 17p + 72$

2) $x^2 + x - 72$

3) $m^2 - 9m + 8$

4) $x^2 + x - 30$

5) $x^2 - 9x - 10$

6) $x^2 + 13x + 40$

7) $ab^2 + 12ab + 32a$

8) $2b^2 - 34b + 140$

9) $w^2 - 7wt - 8t^2$

10) $4a^2b - 28ab + 40b$

6.4 Factoring-Trinomials where $a \neq 1$

Learning Objectives: In this section, you will:

- Factor trinomials with coefficient $a \neq 1$
- Factor trinomials after factoring out the GCF.

When factoring trinomials, we used the ac method to split the middle term and then factor by grouping. The ac method gets it's name from the general trinomial equation, $ax^2 + bx + c$, where a, b, and c are the numbers in front of x^2, x and the constant at the end respectively.
The ac method is named ac because we multiply $a \cdot c$ to find out what we want to multiply to. In the previous lesson we always multiplied to just c because there was no number in front of x^2. This meant the number was 1 and we were multiplying to 1c or just c. Now we will have a number in front of x^2 so we will be looking for numbers that multiply to ac and add to b. Other than this, the process will be the same.

Example A) Factor the trinomial $2x^2 - x - 6$ by the grouping ("ac") method.

This is polynomial of the form ax^2+bx+c. *Determine the value of a, b and c.*

$a=2$, $b=-1$ and $c=-6$

Step1, find "ac": $(2)(-6) = -12$

Step2, find two integers whose product is "ac" and whose sum is "b."

The two integers are -4 and 3.

Step 3, Rewrite the middle term bx as the sum of two terms whose coefficients are integers found in the previous step2.

Rewrite $2x^2 - x - 6$ as

$2x^2-4x+3x-6$

Step 4, Factor by grouping

$2x^2-4x+3x-6$

$2x(x-2) +3(x-2)$

$(x-2)(2x+3)$

Example B) Factor the trinomial $5x^2 + 7x - 6$

$5x^2 + 7x - 6$ a=5, b=7 and c=-6. Determine ac=-30. We need to find two numbers with a product of -30 and a sum of 7. The numbers are -3 and 10, split the middle term.

$5x^2-3x+10x-6$

$x(5x-3) + 2(5x-3)$

$(5x-3)(x+2)$

Example C) Factor the trinomial $3x^2 + 11x + 6$

$3x^2 + 11x + 6$	Multiply to ac or $(3)(6) = 18$, add to 11
$3x^2 + 9x + 2x + 6$	The numbers are 9 and 2, split the middle term.
$3x(x+3) + 2(x+3)$	Factoring by grouping
$(x+3)(3x+2)$	Our solution

When a = 1, or no coefficient in front of x^2, we were able to use a shortcut, using the numbers that split the middle term in the factors. The previous example illustrates an important point, the shortcut does not work when $a \neq 1$. We must go through all the steps of grouping to factor in the problem.

To factor using a GCF, take the greatest common factor (GCF), for the numerical coefficient. When choosing the GCF for the variables, if all terms have a common variable, take the ones with the lowest exponent.

Example E) Factor the trinomial $9x^4 - 3x^3 + 12x^2$

$9x^4 - 3x^3 + 12x^2$	GCF: Coefficients = 3
	Variables (x) = x^2
GCF = $3x^2$	Next, you just divide each monomial by the GCF!
$3x^2(3x^2 - 1x - 4)$	a=3, b=-1 and c=-4 use ac method ac= (3) (-4) =-12

$3x^2(3x^2 - x - 4)$
$3x^2(3x^2 +3x-4x - 4)$ Factor by grouping
$3x^2\{3x(x+1) -4(x +1)\}$
$3x^2(x+1) (3x +1)$

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<u>Worksheet 6.4 Factoring Trinomials where *a≠1*</u>

Factor Completely:

1)$7x^2-48x+36$

2) $7b^2+15b+2$

3) $5a^2-13a-28$

4) $2x^2-5x+2$

5) $2x^2+19x+35$

6) $2b^2-b-3$

7) $5k^2+13k+6$

8) $3x^2-17x+20$

9) $3x^2 + 17xy + 10y^2$

10) $5x^2 + 28xy - 49y^2$

11) $6x^2-39x-21$

12) $21k^2-87k-90$

13) $14x^2-60x+16$

14) $6x^2+29x+20$

15) $4k^2-17k+4$

16) $4x^2+9xy+2y^2$

17) $4m^2-9mn-9n^2$

18) $4x^2 + 13xy + 3y^2$

19) $12x^2 + 62xy + 70y^2$

20) $24x^2 - 52xy + 8y$

6.5 Factoring Special Products

Learning Objectives: In this section, you will:

- Identify and factor special products including a difference of squares, perfect squares, and sum and difference of cubes.
- Factor trinomials after factoring out the GCF.

When we learned how to multiply binomials, we talked about two special products: the Sum and Difference Formula and the Square of a Binomial Formula. Now, we will learn how to recognize and factor these special products.

Factoring the Difference of Two squares:

We use the Sum and Difference Formula to factor the difference of two squares. The difference of two squares is a quadratic polynomial in this form: a^2-b^2. Both terms in the polynomial are perfect squares. In a case like this, the polynomial factors into the sum and difference of the square root of each term. $\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$

Example A) Factor $x^2 - y^2$

$x^2 - y^2$ Subtracting two squares, the square roots are *x* and *y*

(x+ y) (x-y) Our solution

Example B) Factor $x^2 - 16$

$x^2 - 16$ Subtracting two squares, the square roots are *x* and 4

(x+4) (x-4) Our solution

Example C) Factor $9a^2 - 25b^2$

$9a^2 - 25b^2$ Subtracting two squares, the square roots are *3a* and 5b

(3a+5b) (3a-5b) Our solution

Example D) Factor $x^2 + 36$

$x^2 + 36$ No bx term, we use *0x*.

$x^2 + 0x + 36$ Multiply to 36, add to 0: No combinations that multiply to 36 add to 0

Prime, can't factor.

Example E) Factor $x^4 - y^4$

$x^4 - y^4$ Difference of squares with roots x^2 and y^2

$(x^2 + y^2)(x^2 - y^2)$ The first factor is prime, the second is *x* difference of y.

$(x^2 + y^2)(x + y)(x - y)$ Our solution

Perfect Square:

A Perfect square trinomial has the form $\mathbf{a^2 + 2ab + b^2 = (a + b)^2}$ or $\mathbf{a^2 - 2ab + b^2 = (a - b)^2}$

Example A) Factor $x^2 - 6x + 9$

$x^2 - 6x + 9$ Multiply to 9, add to -6

$x^2 - 3x - 3x + 9$ -3 and -3, split the middle term.

$x(x - 3) - 3(x - 3)$ Factoring by grouping

$(x - 3)(x - 3)$

$(x - 3)^2$

Example B) *Let's use the formula.* Factor $x^2 - 6x + 9$

$x^2 - 6x + 9$ Check the first term and the last term are perfect squares.
Check the middle term is twice the product of the square roots of the first and the last terms. This is true also since we can rewrite them.
$x^2 - 2 \cdot 3x + 3^2$ This means we can factor $x^2 - 6x + 9$ as $(x - 3)^2$

Example C) Factor $x^2 + 8x + 16$

$x^2 + 8x + 16$ check the first term and the last term are perfect squares.
Check the middle term is twice the product of the square roots of the first and the last terms. This is true also since we can rewrite them.
$x^2 + 2.4x + 4^2$ This means we can factor $x^2 + 2 * 4x + 16$ as $(x + 4)^2$

Factor a Sum/difference of cubes.

Another factoring shortcut has cubes. With cubes we can either do a sum or a difference of cubes. Both sum and difference of cubes have very similar factoring formulas.

<u>Sum of Cubes</u>:	$\mathbf{a^3 + b^3 = (a + b)\ (a^2 - ab + b^2)}$
<u>Difference of Cubes</u>:	$\mathbf{a^3 - b^3 = (a - b)\ (a^2 + ab + b^2)}$

Example A) Factor $M^3 - 27$

$M^3 - 27$ We have cube roots m and 3.
$(m - 3)\ (m^2 + 3m + 9)$ Use formula

Example B) Factor $x^3 - 8$

$x^3 - 8$
$x^3 - 2^3$ we have cube roots x and 2.
$(x - 2)\ (x^2 + 2x + 2^2)$

Example C) Factor $x^3 - 64$

$x^3 - 64$ we have cube roots x and 4.
$(x - 4)\ (x^2 + 4x + 16)$

Example D) Factor $125p^3+8r^3$

$125p^3+8r^3$ We have cube roots 5p and 2r.
$(5p + 2r)\ (25p^2 - 10r + 4r^2)$ Use formula

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Worksheet 6.5: Factoring Special Products

Factor:

1) $r^2 - 16$

2) $p^2 + 4$

3) $9k^2 - 25$

4) $3x^2 - 27$

5) $16x^2 - 36$

6) $18a^2 - 50b^2$

7) $a^2 - 2a + 1$

8) $x^2 + 6x + 9$

9) $x^2 - 6x + 9$

10) $25p^2 - 10p + 1$

11) $25a^2 + 30ab + 9b^2$

12) $4a^2 - 20ab + 25b^2$

13) $8x^2 - 24xy + 18y^2$

14) $8 - m3$

15) $x^3 - 64$

16) $125a^3 - 64$

17) $64x^3 + 27y^3$

18) $a^4 - 81$

19) $16 - z^4$

20) $m^4 - 81b^4$

6.6 Factoring Strategies

Learning Objectives: In this section, you will:

- Identify and use the correct method to factor various polynomials.

With so many different tools used to factor, it is easy to get lost as to which tool to use when. Here we will attempt to organize all the different factoring types we have seen. A large part of deciding how to solve a problem is based on how many terms are in the problem. For all problem types we will always try to factor out the GCF first.

Factoring Strategy (GCF First)

2-terms: sum or difference of squares or cubes

Difference of Squares	$a^2 - b^2 = (a + b)(a - b)$
Sum of Squares	$a^2 - b^2 = \text{Prime}$
Sum of Cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of Cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

3-terms: 'ac' method, watch for perfect square.

Perfect Square $a^2 + 2ab + b^2 = (a + b)^2$ or $a^2 - 2ab + b^2 = (a - b)^2$

Multiply to 'ac' and add to 'b'

4-terms: grouping method

We will use the above strategies to factor each of the following examples. Here the emphasis will be on which strategy to use rather than the steps used in that method.

Example A) Factor $4x^2 + 56xy + 196y^2$

$4x^2 + 56xy + 196y^2$	GCF first, 4
$4(x^2 + 14xy + 49y^2)$	Three terms, try ac method, multiply to 49, add to 14; 7 and 7, perfect square!
$4(x + 7y)^2$	Our Solution

Example B) Factor $5x^2 y + 15xy - 35x^2 - 105x$

$5x^2 y + 15xy - 35x^2 - 105x$	GCF first, 5x
$5x(xy + 3y - 7x - 21)$	Four terms, try grouping.
$5x[y(x + 3) - 7(x + 3)]$	GCF is $(x + 3)$, factor out $(x+3)$
$5x(x + 3)(y - 7)$	Our Solution

Example C) Factor $100x^2 - 400$

$100x^2 - 400$	GCF first, 100
$100(x\ 2 - 4)$	Two terms, difference of squares
$100(x + 4)(x - 4)$	Our Solution

Example D) Factor $108x^3 y^2 - 39x^2 y^2 + 3xy^2$

$108x^3 y^2 - 39x^2 y^2 + 3xy^2$	GCF first, $3xy^2$
$3xy^2(36x^2 - 13x + 1)$	Three terms, ac method, multiply to 36, add to -13.
$3xy^2(36x^2 - 9x - 4x + 1)$	-9 and -4, split middle term
$3xy^2[9x(4x - 1) - 1(4x - 1)]$	Factor by grouping
$3xy^2(4x - 1)(9x - 1)$	Our Solution

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Worksheet 6.6: Factoring Strategies

Factor:

1) $24az - 18ah + 60yz - 45yh$

2) $5u^2 - 9uv + 4v^2$

3) $-2x^3 + 128y^3$

4) $5n^3 + 7n^2 - 6n$

5) $54u^3 - 16$

6) n 2 − n

7) $x^2 - 4xy + 3y^2$

8) $9x^2 - 25y^2$

9) $m^2 - 4n^2$

10) $36b^2c - 16xd - 24b^2 + 24xc$

11) $128 + 54x^3$

12) $2x^3 + 6x^2y - 20y^2x$

13) $n^3 + 7n^2 + 10n$

14) $27x^3 - 64$

15) $5x^2 + 2x$

16) $3k^3 - 27k^2 + 60k$

17) $mn - 12x + 3m - 4xn$

18) $16x^2 - 8xy + y^2$

19) $27m^2 - 48n^2$

20) $9x^3 + 21x^2 y - 60y^2x$

21) $2m^2 + 6mn - 20n^2$

<u>6.7 Solve by Factoring</u>

Learning Objectives: In this section, you will:

- Solve quadratic equation by factoring and using the zero-product rule.

Zero Product Rule: The zero-product rule tells us that if two factors are multiplied together and the answer is zero, then one of the factors must be zero.

If ab = 0 then either a = 0 or b = 0

Solving Polynomial Equations by factoring and using the zero-product rule

We have learned how to factor quadratic polynomials that are helpful in solving polynomial equations like $ax^2+bx+c=0$. Remember that to solve polynomials in expanded form, we use the following steps:

Step 1: Rewrite the equation in standard form such that: Polynomial expression = 0.
Step 2: Factor the polynomial completely.
Step 3: Use the Zero Product Property to set each factor equal to zero.
Step 4: Solve each equation from step 3.

Example A) Solve $4x^2 + x - 3=0$

Rewrite: No need to rewrite because it is already in the correct form.
Factor:

$4x^2 + x - 3=0$	
$4x^2 - 3x + 4x - 3=0$	The numbers are − 3 and 4, split the middle term
$x(4x - 3)+1(4x - 3)=0$	Factor by grouping
$(4x - 3)(x + 1) =0$	One factor must be zero
$4x - 3=0$ or $x + 1=0$	Set each factor equal to zero
$x = \frac{3}{4}, \quad x = -1$	

Example B) Solve $x^2 = 8x - 15$

$x^2 = 8x - 15$	Rewrite: Set equal to zero by moving terms to the left
$x^2 - 8x + 15 = 0$	Factor using the ac method, multiply to 15, add to – 8
$(x - 5)(x - 3) = 0$	The numbers are − 5 and − 3
$x - 5 =0$ or $x - 3= 0$	Set each factor equal to zero
$x = 5$ or $x = 3$	Our Solution

Example C) Solve $4x^2 = 8x$

$4x^2 = 8x$	Set equal to zero by moving the terms to left
$4x^2 - 8x = 0$	Factor out the GCF of $4x$
$4x(x - 2)= 0$	One factor must be zero
$4x = 0$ or $x - 2= 0$	Set each factor equal to zero
$x = 0$ or 2	Our Solution

Example D) Solve $2x^3 - 14x^2 + 24x = 0$

$2x^3 - 14x^2 + 24x = 0$	Factor out the GCF of $2x$
$2x\ (x\ 2 - 7x + 12) = 0$	Factor with ac method, multiply to 12, add to – 7
$2x\ (x - 3)(x - 4) = 0$	The numbers are − 3 and – 4
$2x = 0$ or $x - 3 = 0$ or $x - 4 = 0$	Set each factor equal to zero
$x = 0$ or $x = 3$ or $x = 4$	Our Solutions

Example E) Solve $x^2 + 4x = 5$

$x^2 + 4x - 5 = 0$

$(x + 5)\ (x - 1) = 0$

$x + 5 = 0 \quad x - 1 = 0$

$x = -5 \quad$ or $\quad x = 1$

Example F) Solve $x^2 - 81 = 0$

$(x+9)\ (x-9) = 0$

$x+9=0$ or $x-9=0$

$x=9$ or $x=-9$

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Worksheet 6.7: Solve by Factoring

Solve:

1) $(k-7)\ (k+2)=0$

2) $(x-1)\ (x+4)=0$

3) $6x^2 - 150 = 0$

4) $2n^2+10n-28=0$

5) $7x^2+26x+15=0$

6) $5n^2 - 9n - 2 = 0$

7) $x^2-4x-8=-8$

8) $x^2-5x-1=-5$

9) $49p^2 + 371p - 163 = 5$

10) $7x + 17x - 20 = -8$

11) $7r^2 + 84 = -49r$

12) $x^2 - 6x = 16$

13) $3v^2 + 7v = 40$

14) $35x^2 + 120x = -45$

15) $4k^2+18k - 23 = 6k - 7$

16) $9x^2 - 46+7x = 7x + 8x^2+3$

17) $2m^2+19m + 40 = -2m$

18) $40p^2 + 183p - 168 = p + 5p$

7.1 Reduce Rational Expressions

Learning Objectives: In this section, you will:

- Reduce ration expressions by dividing out common factors.

Rational Expression: Any expression that can be written as the quotient of two polynomials.

$$\frac{7x^2-3x+5}{2x-1} \qquad \frac{3}{7} \qquad \frac{a+b}{a-b}$$

Remember that for a fraction, the denominator cannot be zero, so any values we substitute in for our variables cannot make the denominator zero.

If we are given a value, we just substitute that value in for our variable to simplify the expression.

To reduce a fraction, we know we remove what they have in common. This is the same procedure done for these new rational expressions since they are really the same idea as every fraction you have ever reduced. The key is to remember that to do this, the problem must be factored first.

Point: To reduce a ration expression, we must first factor and then remove what the numerator and denominator have in common

Example A) Evaluate $\frac{a+2}{a^2-2a+4}$ when $a=-1$

$\frac{(-1)+2}{(-1)^2-2(-1)+4}$ Substitute -1 for a

$\frac{-1+2}{1+2+4}=\frac{-1}{7}$ Use order of operations to simplify

Example B) Simplify $\frac{14a^2}{21a^5}$

$\frac{(2\times 7)a^2}{(3\times 7)a^5}$ Factor the numbers

$\frac{(2\times \cancel{7})\cancel{a^2}}{(3\times \cancel{7})\cancel{a^5}a^3}=\frac{2}{3a^3}$ Cross out the common factors

Example C) Determine any excluded values $\frac{y-1}{12y^2-24y}$

$12y^2-24y=12y(y-2)$ Factor the denominator

$12y(y-2)=0$ Set the denominator equal to zero to find excluded values

$12y=0\ or\ y-2=0$ Set each factor equal to zero

$y=0\ or\ y=2$ Our solution

Example D) Simplify $\frac{x^2-3x+2}{x^2+5x-14}$

$x^2 - 3x + 2 = (x-2)(x-1)$ Factor the numerator

$x^2 + 5x - 14 = (x-2)(x+7)$ Factor the denominator

$\frac{x^2-3x+2}{x^2+5x-14} = \frac{(x-2)(x-1)}{(x-2)(x+7)}$ Rewrite the fraction in factored form

$\frac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}(x+7)} = \frac{x-1}{x+7}$ Cross out the common factor

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<u>Worksheet 7.1: Reduce Rational Expressions</u>

Evaluate:

1) $\frac{n+2}{3n}$ when $n = -5$

2) $\frac{2x^2-x+7}{x+3}$ when $x = 2$

3) $\frac{7p-2}{8p}$ when $p = 3$

State the excluded values for each:

4) $\frac{7n}{2}$

5) $\frac{4k-10}{2k^2+3k+1}$

6) $\frac{12p}{25p^2-5p}$

Simplify the expression:

7) $\frac{18x^6}{27x^4}$

8) $\frac{10a^3b}{-15ab^3}$

9) $\frac{-16x^2y^7z}{12x^5y^3z^4}$

10) $\frac{2x+6}{4x-12}$

11) $\frac{x^2+9x+20}{2x+8}$

12) $\frac{3x+18}{x^2+6x}$

13) $\frac{4x+4}{x^2+4x+3}$

14) $\frac{a^2-b^2}{a^2+2b+b^2}$

15) $\frac{2x^2-x-3}{x^2-1}$

16) $\frac{50v-80}{50v+20}$

17) $\frac{n-9}{9n-81}$

18) $\frac{18x-24}{60}$

19) $\frac{14}{81y^3+36y^2}$

20) $\frac{56x-48}{24x^2+56x+32}$

21) $\frac{7x^2-32x+16}{4x-16}$

7.2 Multiply and Divide Rational Expressions

Learning Objectives: In this section, you will:

- Multiply and divide rational expressions.

The process of multiplying or dividing rational expressions is the same process we have used for fractions in the past. The only thing we need to do is factor the problem first before we actually multiply.

Point: Remember all division is just multiplying by the reciprocal. Change the problem to multiplication first before factoring and simplifying.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

Point: We can cancel any numerator factor with any denominator factor. This is called cross cancelling.

Example A) Multiply: $\frac{2a^2}{b^3} \cdot \frac{ab}{6}$

$\frac{\cancel{2}a^2}{b^3} \cdot \frac{ab}{\cancel{6}3}$ Cross cancel any numbers

$\frac{a^2}{b^{\cancel{3}2}} \cdot \frac{a\cancel{b}}{3}$ Cross cancel any variables

$\frac{a^3}{3b^2}$ Multiply across to get the final solution

Example B) Divide $\frac{7x^2}{12y} \div \frac{14x^3}{48y^3}$

$\frac{7x^2}{12y} \cdot \frac{48y^3}{14x^3}$ Rewrite the problem as multiplication

$\frac{1\cancel{7}x^2}{1\cancel{12}y} \cdot \frac{4\cancel{48}y^3}{2\cancel{14}x^3}$ Cross cancel any numbers

$\frac{\cancel{x^2}}{\cancel{y}} \cdot \frac{4y^{\cancel{3}2}}{2x^{\cancel{3}}}$ Cross cancel variables

$\frac{1}{1} \cdot \frac{4y^2}{2x} = \frac{2y^2}{x}$ Multiply across and simplify any remaining numbers

Example C) Multiply $\frac{6}{x^2+9x+20} \cdot \frac{2\text{x}+10}{6\text{x}-12}$

$\frac{6}{(\text{x}+4)(\text{x}+5)} \cdot \frac{2(\text{x}+5)}{6(\text{x}-2)}$ Factor each numerator and denominator

$\frac{\cancel{6}}{(\text{x}+4)\cancel{(\text{x}+5)}} \cdot \frac{2\cancel{(\text{x}+5)}}{\cancel{6}(\text{x}-2)}$ Cross cancel

$\frac{2}{(\text{x}+4)(\text{x}-2)}$ Multiply across to get the solution

Example D) Divide $\frac{x^2+5x-24}{2x+2} \div \frac{(3x+24\}}{x^2-8x-9}$

$\frac{x^2+5x-24}{2x+2} \cdot \frac{x^2-8x-9}{3x+24}$ Rewrite as multiplication

$\frac{(x+8)(x-3)}{2(x+1)} \cdot \frac{(\text{x}-9)(\text{x}+1)}{3(\text{x}+8)}$ Factor each numerator and denominator

$\frac{\cancel{(x+8)}(x-3)}{2\cancel{(x+1)}} \cdot \frac{(\text{x}-9)\cancel{(\text{x}+1)}}{3\cancel{(\text{x}+8)}}$ Cross cancel

$\frac{(x-3)(x-9)}{6}$ Multiply across to get the solution

===

Worksheet 7.2: Multiply and Divide Rational Expressions

Multiply or Divide:

1) $\frac{24x^3}{25y^5} \cdot \frac{15y^2}{8x^2}$

2) $\frac{4a^3}{b^2} \div \frac{3a}{4b^4}$

3) $\frac{24x^3}{50x} \cdot \frac{30}{8x^2}$

4) $\frac{5x-15}{4x^2} \cdot \frac{x^3}{6x-18}$

5) $\frac{4x}{8x+8} \cdot \frac{x^2+8x+7}{8x^3}$

6) $\frac{x^2-5x-6}{5x+15} \div \frac{x^2-3x-4}{7x+21}$

7) $\frac{6x+24}{5x-35} \cdot \frac{9x-63}{7x+28}$

8) $\frac{3x-21}{x^2-3x-28} \div \frac{2x+8}{5x+20}$

9) $\frac{y^2-9}{y+3} \cdot \frac{2y-3}{y-3}$

10) $\frac{2z^2+3z-2}{(2z-1)^2} \cdot \frac{2z}{z+2}$

11) $\frac{9x^2+19x+2}{4-x^2} \div \frac{9x^2-8x-1}{x^2-4x+4}$

12) $\frac{a+9}{3a+1} \cdot \frac{3}{9+a}$

13) $\frac{x^2+7x+10}{x^2+4x+4} \div \frac{1}{x^2-4}$

14) $\frac{x^2-y^2}{y+x} \div (y-x)^2$

15) $\frac{1}{x^2} \cdot \frac{x-1}{x+3} \div \frac{x-1}{x^3}$

16) $\frac{8}{r^2+7r+6} \div \frac{8}{8r+8}$

17) $\frac{2}{x-6} \cdot \frac{x-2}{x^2-8x+12}$

18) $\frac{1}{n+7} \cdot \frac{8n^2+56n}{8}$

19) $\frac{y^2-2y+1}{y+1} \div \frac{6y-6}{y+1}$

20) $\frac{2a}{2a-4} \cdot (2a^2-20a+32)$

21) $\frac{27r-63}{3r-7} \div (r-6)$

7.3 Least Common Denominators

Learning Objectives: In this section, you will:

- Find the least common denominator for rational functions

Least Common Denominator: The smallest denominator that all other denominators can divide into.

This is the same idea as a least common multiple, just in the denominator of a term. Since rational expressions are just fractions, we will find the least common denominator(LCD) very similarly to what we have always done. Often, a tree diagram can be helpful for numerical values.

Step 1:	Factor the denominators completely
Step 2:	Looking at each denominator separately, count up how many of a given factor it has
Step 3:	Compare the factor count to all other factored denominators, whoever has the most of that factor, that is how many copies to take. We do not just take all the copies added up
Step 4:	Repeat this process until you have went through all different terms

Whatever you have at the end is the LCD.

Point: We want the most a term appears in any of the denominators. If a term appears the same amount in all denominators, that is the most so we will take that amount.

Working with variables becomes easier when you see that x+1 has no x's in it. Remember x and x+1 might as well be x and y. If x=2, then x+1=3, which have nothing in common. Anytime a variable has a number added or subtracted from it, it is an entirely different variable.

Example A) Find the LCD: $15x^2y$ and $12x^2y^3$

$3 \cdot 5 \cdot x^2y$ and $2^2 \cdot 3 \cdot x^2y^3$	Factor each denominator fully
$3 \cdot 5 \cdot 2^2$	Notice both numbers require one 3, the first number requires a 5 and the second number requires two 2s, so this gives us the numerical LCD
x^2y^3	Notice both terms require two x's, so we take two. The first term needs one y while the other needs three, so we take three.
$60x^2y^3$	Solution

Example B) Find the LCD: $5x+5$ and 25x

$5(x+1)$ and $5^2 \cdot x$	Factor each denominator fully
5^2	Notice the first term requires one 5, while the other requires two 5, so we take the greater amount
$x(x+1)$	Notice the first term requires an x+1, while the second requires an x, so we take each
$25x(x+1)$	Solution

Example C) Rewrite both fractions over their LCD $\frac{2}{x+1}, \frac{3x}{x-5}$

$(x+1)(x-5)$	Find the LCD
$\frac{2}{x+1} \cdot \frac{x-5}{x-5}, \frac{3x}{x-5} \cdot \frac{x+1}{x+1}$	Multiply each fraction by its missing factors
$\frac{2x-10}{(x+1)(x-5)}, \frac{3x^2+3x}{(x+1)(x-5)}$	Multiply out the numerators

Example D)) Rewrite both fractions over their LCD $\frac{x+2}{x^2-7x-12}, \frac{5}{x^2-5x+6}$

$x^2-7x+12=(x-3)(x-4)$	Factor the first denominator
$x^2-5x+6=(x-2)(x-3)$	Factor the second denominator
$(x-3)(x-4)(x-2)$	Find the LCD
$\frac{x+2}{(x-3)(x-4)} \cdot \frac{x-2}{x-2}, \frac{5}{(x-2)(x-3)} \cdot \frac{x-4}{x-4}$	Multiply each fraction by its missing factors
$\frac{x^2-4}{(x-3)(x-4)(x-2)}, \frac{5x-20}{(x-3)(x-4)(x-2)}$	Multiply out the numerators

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Worksheet 7.3: Least Common Denominators

Find the LCD:

1) $3x^2y^4$ and $6xy^6$

2) x and $x+5$

3) x^2-25 and $x-5$

4) $x^2+3x+2,\ x^2+5x+6$

5) $4y-8, y-2, 4$

6) x^2+2x+1, x^2+3x+2

Write the fractions over their common denominator:

7) $\frac{3a}{5b^2}, \frac{1}{10a^3b}$

8) $\frac{3x+1}{x^2-11x+30}, \frac{7}{x-6}$

9) $\frac{x+2}{x-4}, \frac{x-4}{x+2}$

10) $\frac{5}{y^2-4y}, \frac{3}{y}, \frac{-2}{y-4}$

11) $\frac{5}{x^3}, \frac{x-3}{x^2+2x}$

12) $\frac{4x}{x^2-x-6}, \frac{x-3}{x+2}$

13) $\frac{z}{z^2-25}, \frac{2z}{z^2-10z+25}$

14) $\frac{1}{5x^3y^2z}, \frac{2x}{y^3z}$

15) $\frac{7}{x^2+3x+2}, \frac{1}{x^2+5x+6}$

16) $\frac{2x+1}{x^2-x-12}, \frac{3x+2}{x^2+4x+3}$

17) $\frac{x+9}{3x}, \frac{2x}{x-5}$

18) $\frac{4}{3y^2-6y}, \frac{y^2}{y^2+5x-14}$

19) $\frac{7}{x^2-4}, \frac{6}{x^2+4x+4}, \frac{8}{x^2-4x+4}$

20) $\frac{2x+3}{4x^2+4x+1}, \frac{7x}{6x^2+x-3}$

21) $\frac{8}{5x}, \frac{x-2}{6x^2-6} \frac{5x}{3x-3}$

7.4 Adding and Subtraction Rational Expressions

Learning Objectives: In this section, you will:

- Adding and subtracting rational expressions by having a least common denominator

Adding or subtracting rational expressions is the same process as when you first learned to add and subtractions in primary school. We need a common denominator, then we add or subtract the numerators and keep the denominator what it is.

Remember we change the numerator when combining fractions with addition and subtraction, not the denominator.

If you need practice finding the least common denominator(LCD), please review section 7.3.

Point: We simplify fractions after we have done the addition or subtraction. If we simplify after finding the LCD but before combining the fractions, we will be back to where we started and cannot combine them.

A few additional things to keep in mind. Once we combine our fractions through addition or subtraction, we do not multiply out the denominator. The reason is that if we can simply the final solution, we will need it factored to do that simplification. We do simplify, ie. distribute and collect like terms for the numerator. Once done, we factor the numerator if it can simplify with a factor in the denominator. If not, we do not need to factor the numerator.

Example A) Add $\frac{2}{3x} + \frac{5}{7x^2}$

LCD: $21x^2$	Find the LCD for every denominator
$\frac{2}{3x} \cdot \frac{7x}{7x} + \frac{5}{7x^2} \cdot \frac{3}{3}$	Multiply each fraction by the factor needed to put them over the LCD
$\frac{14x}{21x^2} + \frac{15}{21x^2}$	Combine terms after multiplying to get LCD
$\frac{14x+15}{21x^2}$	Add the numerators together while writing the fraction over the LCD.

Example B) Subtract $\frac{3}{2x+4} - \frac{x}{x^2-4}$

$2x + 4 = 2(x + 2)$

$x^2 - 4 = (x + 2)(x - 2)$ Factor the denominators

LCD: 2(x+2)(x-2) Find the LCD for every denominator

$\frac{3}{2(x+1)} \cdot \frac{x-2}{x-2} - \frac{x}{(x-2)(x+2)} \cdot \frac{2}{2}$ Multiply each fraction by the factor needed to put them over the LCD

$\frac{3(x-2)}{2(x-2)(x+2)} - \frac{2x}{2(x-2)(x+2)}$ Combine terms after multiplying to get LCD

$\frac{3x-6-2x}{2(x-2)(x+2)}$ Add the numerators together

$\frac{x-6}{2(x-2)(x+2)}$ Simplify

Example C) Add $\frac{x+3}{x^2-4x+3} + \frac{x-1}{x^2-2x-3}$

$x^2 - 4x + 3 = (x - 3)(x - 1)$

$x^2 - 2x - 3 = (x - 3)(x + 1)$ Factor each denominator

LCD: $(x - 3)(x - 1)(x + 1)$ Find the LCD for every term

$\frac{x+3}{(x-3)(x-1)} \cdot \frac{x+1}{x+1} + \frac{x-1}{(x-3)(x+1)} \cdot \frac{x-1}{x-1}$ Multiply each fraction by the factor needed to put them over the LCD

$\frac{x^2+4x+3}{(x-3)(x-1)(x+1)} + \frac{x^2-2x+1}{(x-3)(x-1)(x+1)}$ Multiply out the numerators

$\frac{2x^2+2x+4}{(x-3)(x-1)(x+1)}$ Add the numerators together

Example D) Add and subtract $\frac{x}{2} + \frac{1}{y} - \frac{x^2}{x-y}$

LCD: $2y(x - y)$ Find the LCD

$\frac{x}{2} \cdot \frac{y(x-y)}{y(x-y)} + \frac{1}{y} \cdot \frac{2(x-y)}{2(x-y)} - \frac{x^2}{x-y} \cdot \frac{2y}{2y}$ Multiply each fraction by the factor needed to put them over the LCD

$\frac{x^2y-xy^2}{2y(x-y)} + \frac{2x-2y}{2y(x-y)} - \frac{2x^2y}{2y(x-y)}$ Multiply out the numerators

$\frac{-x^2y-xy^2+2x-2y}{2y(x-y)}$ Add the numerators together

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Worksheet 7.4: Adding and subtracting rational expressions

Evaluate:

1) $\frac{1}{2x}+\frac{3}{5y}$

2) $\frac{4}{3x^2}-\frac{7}{6x}$

3) $\frac{1}{x}+\frac{2}{x^2y}-\frac{x}{y^2}$

4) $\frac{2}{x^2+6x+5}+\frac{9}{x^2+5x+4}$

5) $\frac{x-1}{x+1}-\frac{x+1}{x-1}$

6) $\frac{y}{y^2-9}+\frac{2}{y^2-6x+9}$

7) $\frac{3z}{z^2-5x-14}-\frac{z+2}{z^2-8z+7}$

8) $\frac{5}{x-2}+\frac{x+5}{x-2}$

9) $\frac{7}{4x^2y}+\frac{3}{20xy^3}$

10) $4y+\frac{y+9}{5y}$

11) $\frac{y}{y+5}-\frac{6}{y-2}$

12) $\frac{4}{x-7}+\frac{x+2}{7-x}$

13) $y-\frac{1}{x}-\frac{x^2}{y-x}$

14) $\frac{2}{x^2-5x+4}+\frac{-2}{x^2-4}$

15) $\frac{7}{x}+\frac{11}{2x^2}$

16) $\frac{6x-3}{x^2-6x-7}+\frac{4-5x}{x^2-6x-7}$

17) $\frac{2}{y+1}-\frac{y}{y-2}+\frac{y^2+3}{y^2-y-2}$

18) $\frac{1}{x^2+5x+6}-\frac{2}{x^2+3x+2}+$

$\frac{1}{x^2-3x-4}$

19) $\frac{3x-2}{x^2-4}-\frac{x+4}{x^2-4x+4}$

20) $\frac{x^2-2x+3}{x^2+7x+12}-\frac{x^2-4x-5}{x^2+7x+12}$

21) $2-\frac{x+3}{(2x+1)(x-7)}$

7.5 Complex Fractions

Learning Objectives: In this section, you will:

- Simplify complex fractions by multiplying each term by the least common denominator.

Definition: A complex fraction is a fraction that has fractions in the numerator and/or denominator.

As with all rational expressions, we need the problem to be factored before we work with it. Once it is factored, our goal is to multiply every term by the least common denominator.

Method

1. Find the LCD of every term.
2. Multiply every term by the LCD.
3. Reduce the fraction.
4. Simplify(distribute and collect like terms).

This works because if we multiply every term by the LCD, we are multiplying by a term over itself, which means we are multiplying by 1. This method does not work for adding or subtracting fractions.

Point: When this method is used correctly, every denominator will cancel away with the numerator. This removes all the denominators and gives us a problem we know now to solve.

Example A) Simplify $\frac{5}{\frac{3}{5}+\frac{2}{7}}$

$LCD: 35$	Find the LCD
$\frac{5\cdot 35}{\left(\frac{3}{5}\right)35+\left(\frac{2}{7}\right)35}$	Multiply every term by the LCD
$\frac{175}{3(7)+2(5)}=\frac{175}{31}$	Simplify

Example B) Simplify $\frac{\frac{3}{x+4}-2}{4+\frac{2}{x+4}}$

LCD: x+4	Find the LCD for every denominator
$\frac{\left(\frac{3}{x+4}\right)(\mathrm{x}+4)-(2)(\mathrm{x}+4)}{4(\mathrm{x}+4)+\left(\frac{2}{x+4}\right)(\mathrm{x}+4)}$	Multiply every term by the LCD
$\frac{\left(\frac{3}{\cancel{x+4}}\right)\cancel{(\mathrm{x}+4)}-(2)(\mathrm{x}+4)}{4(\mathrm{x}+4)+\left(\frac{2}{\cancel{x+4}}\right)\cancel{(\mathrm{x}+4)}}$	Reduce fractions
$\frac{3-2x-8}{4x+16+2}=\frac{-2\mathrm{x}-5}{4\mathrm{x}+18}$	Simplify

Example C) Simplify $\dfrac{1-\frac{1}{x^2}}{1-\frac{1}{x}}$

LCD: x^2 — Find the LCD for every denominator

$$\frac{1(x^2)-\left(\frac{1}{x^2}\right)(x^2)}{1(x^2)-\left(\frac{1}{x}\right)(x^2)}$$ Multiply each fraction by the LCD

$$\frac{1(x^2)-\left(\frac{1}{\cancel{x^2}}\right)(\cancel{x^2})}{1(x^2)-\left(\frac{1}{\cancel{x}}\right)(x^{\cancel{2}})}$$ Reduce fractions

$$\frac{x^2-1}{x^2-x}$$ Multiply

$$\frac{(x-1)(x+1)}{x(x-1)}$$ Factor to find like terms to simplify

$$\frac{\cancel{(x-1)}(x+1)}{x\cancel{(x-1)}}=\frac{x+1}{x}$$ Reduce the fraction

Example D) Simplify $\dfrac{\frac{1}{x-3}+\frac{x}{x+4}}{\frac{2x}{x+4}-\frac{1}{x^2+x-12}}$

LCD: $(x+4)(x-3)$ — Find the LCD for every denominator

$$\frac{\left(\frac{1}{x-3}\right)(x+4)(x-3)+\left(\frac{x}{x+4}\right)(x+4)(x-3)}{\left(\frac{2x}{x+4}\right)(x+4)(x-3)-\left(\frac{1}{x^2+x-12}\right)(x+4)(x-3)}$$ Multiply every term by the LCD

$$\frac{\left(\frac{1}{\cancel{x-3}}\right)(x+4)\cancel{(x-3)}+\left(\frac{x}{\cancel{x+4}}\right)\cancel{(x+4)}(x-3)}{\left(\frac{2x}{\cancel{x+4}}\right)\cancel{(x+4)}(x-3)-\left(\frac{1}{\cancel{(x+4)(x-3)}}\right)\cancel{(x+4)(x-3)}}$$ Reduce fractions

$$\frac{x+4+x^2-3x}{2x^2-6x-1}=\frac{x^2-2x+4}{2x^2-6x-1}$$ Multiply and simplify

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Worksheet 7.5: Complex Fractions

Solve:

1) $\dfrac{\frac{4}{7}}{\frac{1}{49}+\frac{3}{2}}$

2) $\dfrac{\frac{2}{m-1}}{\frac{1}{m}-\frac{1}{m-1}}$

3) $\dfrac{\frac{x}{6}+\frac{2}{x^2}}{2}$

4) $\dfrac{\frac{1}{y^2}-1}{1+\frac{1}{y}}$

5) $\dfrac{2-\frac{3}{x+3}}{5+\frac{10}{x+3}}$

6) $\dfrac{\frac{n}{4n+3}}{\frac{n+1}{16n^2-9}}$

7) $\dfrac{\frac{1}{y^2}-\frac{1}{xy}-\frac{2}{x^2}}{\frac{1}{y^2}-\frac{3}{xy}+\frac{2}{x^2}}$

8) $\dfrac{\frac{x}{x+3}-\frac{x}{x+3}}{\frac{x}{x+3}+\frac{x}{x-3}}$

9) $\dfrac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{a^2}+\frac{1}{b^2}}$

10) $\dfrac{\frac{2}{x-4}-\frac{6}{x+2}}{\frac{1}{x+2}+\frac{8}{x-4}}$

11) $\dfrac{\frac{4}{x+9}-\frac{6}{x+3}}{\frac{x-4}{x+9}}$

12) $\dfrac{\frac{y}{y+5}+\frac{3}{y+2}}{\frac{2}{y+2}-\frac{y}{y+5}}$

13) $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$

14) $\dfrac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

15) $\dfrac{\frac{y^2-y-2}{y^2-3y-4}}{\frac{y^2+4y+3}{y^2-y*12}}$

16) $\dfrac{x+7+\frac{3}{x+2}}{x-9+\frac{5}{x+2}}$

17) $\dfrac{x-\frac{9}{x}}{2+\frac{6}{x}}$

18) $\dfrac{\frac{4}{a-b}-\frac{3}{a+b}}{\frac{8}{a^2-b^2}}$

19) $\dfrac{1-\frac{1}{x}}{x-2+\frac{1}{x}}$

20) $\dfrac{\frac{1}{3x}+\frac{2}{x}}{\frac{4}{3}+\frac{1}{x}}$

21) $\dfrac{\frac{y}{y-3}+\frac{4}{y}}{3+\frac{1}{3-y}}$

7.7 Solving Rational Equations

Learning Objectives: In this section, you will:

- Solve rational equations by multiplying by the LCD

The toughest part of solving a rational equation is having fractions. If we can remove the fractions, we would have the sort of problems we have solved previously. Our technique will remove the denominator in order to give an easier problem to solve.

If the problem is a simple fraction=fraction, we can cross multiply:

$$\frac{a}{b} = \frac{c}{d}\, becomes \text{ ad} = \text{bc}$$

For more complicated problems we will follow this method

1. Find the LCD of every term.
2. Multiply every term by the LCD.
3. Solve the resulting problem.
4. Check if any of the solutions make a denominator zero. If so, we must remove that solution.

Remember to check the solution here. Since we are not solving the original problem but instead we are solving a similar problem, its possible to get solutions that do not work because they would make us divide by zero. These types of solutions are called extraneous solutions. We never loose solutions, but sometimes gain ones we cannot use.

Point: When this method is used correctly, every denominator will cancel away with the numerator. This removes all the denominators and gives us a problem we know now to solve.

This method works since we are allowed to multiply both sides of an equation by the same term. This does not work in adding and subtraction rationals.

Example A) Add $\frac{3}{2x} = \frac{5}{7}$

$3(7) = 5(2x)$ Cross multiply

$21 = 10x$ Simplify

$x = \frac{21}{10}$ Solve and check

Example B) Subtract $\frac{2x}{3}+\frac{5x}{7}=\frac{3}{14}$

LCD: 42 — Find the LCD for every denominator

$\frac{2x}{3}\cdot\frac{42}{1}+\frac{5x}{7}\cdot\frac{42}{1}=\frac{3}{14}\cdot\frac{42}{1}$ — Multiply each fraction by the LCD

$\frac{2x}{\cancel{3}}\cdot\frac{\cancel{42}\ 14}{1}+\frac{5x}{\cancel{7}}\cdot\frac{\cancel{42}\ 6}{1}=\frac{3}{\cancel{14}}\cdot\frac{\cancel{42}\ 3}{1}$ — Cross cancel individual terms

$28x+30x=9$ — Multiply

$x=\frac{9}{58}$ — Solve and check

Example C) Solve $\frac{3}{x+2}-\frac{1}{x}=\frac{1}{5x}$

LCD: $5x(x+2)$ — Factor each denominator

$\frac{3}{x+2}\cdot\frac{5x(x+2)}{1}-\frac{1}{x}\cdot\frac{5x(x+2)}{1}=\frac{1}{5x}\cdot\frac{5x(x+2)}{1}$ — Multiply each fraction by the LCD

$\frac{3}{\cancel{x+2}}\cdot\frac{5x\cancel{(x+2)}}{1}-\frac{1}{\cancel{x}}\cdot\frac{5\cancel{x}(x+2)}{1}=\frac{1}{\cancel{5x}}\cdot\frac{\cancel{5x}(x+2)}{1}$ — Cross cancel individual terms

$15x-5(x+2)=x+2$ — Multiply

$10x-10=x+2$ — Simplify

$x=\frac{4}{3}$ — Solve and check

Example D) Add and subtract $\frac{2}{x+3} - \frac{3x+5}{x^2+4x+3} = \frac{5}{x+1}$

LCD: $(x+1)(x+3)$ — Find the LCD for every denominator

$$\frac{2}{x+3} \cdot \frac{(x+1)(x+3)}{1} - \frac{3x+5}{(x+1)(x+3)} \cdot \frac{(x+1)(x+3)}{1} = \frac{5}{x+1} \cdot \frac{(x+1)(x+3)}{1}$$

Multiply every term by the LCD

$$\frac{2}{\cancel{x+3}} \cdot \frac{(x+1)\cancel{(x+3)}}{1} - \frac{3x+5}{\cancel{(x+1)(x+3)}} \cdot \frac{\cancel{(x+1)(x+3)}}{1} = \frac{5}{\cancel{x+1}} \cdot \frac{\cancel{(x+1)}(x+3)}{1}$$

Cross cancel

$2(x+1) - (3x+5) = 5(x+3)$ — Multiply

$-x - 3 = 5x + 15$ — Simplify

$\cancel{x = -3}$, no solution — Solve and Check

Notice that -3 makes two of the denominators zero, so we cannot include it.
That means there is no solution.

==

Worksheet 7.7: Solving Rational Equations

Solve:

1) $\frac{2z}{3} = \frac{z+1}{4}$

2) $\frac{3x+4}{7x-2} = \frac{2x-3}{7x-2}$

3) $\frac{x-3}{7} = \frac{2x+11}{4}$

4) $\frac{1}{6x^2} = \frac{1}{3x^2} - \frac{1}{x}$

5) $\frac{y^2}{y-2} = \frac{4}{y-2}$

6) $\frac{6}{x+1} = \frac{x}{x-1}$

7) $\frac{1}{x-4} + \frac{x}{x-2} = \frac{2}{x^2-6x+8}$

8) $\frac{2x}{x^2-9} + \frac{3}{x+3} = \frac{1}{x-3}$

9) $\frac{3x+4}{x-8} - \frac{2}{8-x} = 1$

10) $\frac{x}{3x+1} = \frac{4}{3x+1} = 6$

11) $\frac{y+5}{y+8} = 1 + \frac{6}{y+1}$

12) $\frac{5}{x^3+5x^2} = \frac{4}{x+5} + \frac{1}{x^2}$

13) $\frac{n+5}{n^2+n} - \frac{1}{n^2+n} = \frac{n-6}{n+1}$

14) $\frac{1}{x-2} + \frac{1}{x^2-7x+10} = \frac{6}{x-2}$

15) $\frac{6}{v^2-2v-3} - \frac{1}{v^2-1} = \frac{2}{v^2-4v+3}$

16) $\frac{2x}{x-3} - \frac{6}{x} = \frac{18}{x^2-3x}$

17) $\frac{3}{9} + \frac{1}{2x} = \frac{1}{3}$

18) $\frac{7}{x-2} - \frac{8}{x+5} = \frac{1}{2x^2+6x-20}$

19) $\frac{-1}{3} - \frac{4}{9v} = \frac{4}{9} - \frac{1}{6v}$

20) $\frac{5}{2x-1} - \frac{7}{3x+2} = \frac{9}{6x^2+x-2}$

<u>8.1</u> <u>Square Roots and 8.2 Higher Roots</u>

Learning Objectives: In this section, you will:

- Simplify radical expressions.

Square Root ($\sqrt{\ }$): The square root of a number is a value that, when multiplied by itself, gives the original number.

Example A) Find $\sqrt{9}$

It represents the square root of 9, which is 3 because 3 multiplied by itself 3 * 3 equals 9.

Cube Root ($\sqrt[3]{\ }$): The cube root of a number is a value that, when multiplied by itself twice (cubed), gives the original number. Follow similar procedures for higher roots, such as 4th root, etc.

Example B) Find $\sqrt[3]{8}$

It represents the cube root of 8, which is 2 because 2 multiplied by itself twice 2 * 2 * 2 equals 8.

Simplify radical means remove perfect square factors: If the number under the radical sign (the radicand) has perfect square factors, you can simplify by taking the square root of those factors out of the radical.

Example C) Simplify $\sqrt{16}$

Recognize that 16 is a perfect square, and its square root is 4. Result: $\sqrt{16} = 4$

Or

Prime factorization of 16 = 2^4, $\sqrt{16} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2} = 2 \cdot 2 = 4$

Example D) Simplify $\sqrt[3]{64}$

Recognize that 64 is a perfect cube, and its cube root is 4. Result: $\sqrt[3]{64} = 4$

Or

Sometimes it is not easy to see the perfect cubes, in this case find the prime factorization of 64 = 2^6, $\sqrt[3]{64} = \sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = 2 \cdot 2 = 4$

Example E) Simplify $\sqrt{50}$

Step 1: Recognize that 50 is not a perfect square, but it contains a perfect square factor, which is 25.

Step 2: Rewrite 50 as 25*2.

Step 3: Take the square root of 25, which is 5. Result: $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$

Or

Use prime factorization if you cannot find perfect square: $\sqrt{50} = \sqrt{5 \cdot 5 \cdot 2} = 5\sqrt{2}$

In prime factorization it is easy to see which identical two numbers' product is a perfect square (that is the number that will move outside of the square root).

Example F) Simplify $\sqrt[3]{48}$

Step 1: Recognize that 48 is not a perfect cube, but it contains a perfect cube factor, which is 8.

Step 2: Rewrite 48 as 8 * 6.

Step 3: Take the cube root of 8, which is 2. Result: $\sqrt[3]{48}$ = 2 $\sqrt[3]{6}$

Or

Use prime factorization if you cannot find perfect square: $\sqrt[3]{48}$ = $\sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3}$ =
= 2 $\sqrt[3]{2 \cdot 3}$ =2 $\sqrt[3]{6}$ (in this case look for the product of three identical numbers - perfect cube - and that number will move outside of the cube root)

Example G) Simplify $\sqrt{x^4}$

Use same strategy with variables as with numbers: use perfect squares or prime factorization: $\sqrt{x^4} = \sqrt{x \cdot x \cdot x \cdot x} = x^2$

Example H) Simplify $\sqrt[4]{32a^5b^8}$

$$\sqrt[4]{32a^5b^8} = \sqrt[4]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot a \cdot a \cdot a \cdot a \cdot a \cdot b^4 \cdot b^4} = 2ab^2\sqrt[4]{2a}$$

==

<u>Worksheet: 8.1 Square Roots and 8.2 Higher Roots</u>

1) Evaluate: $\sqrt{64}$

2) Simplify: $\sqrt{75}$

3) Simplify: $4\sqrt{72}$

4) Simplify: $\sqrt[3]{54}$

5) Simplify: $-8\sqrt{600}$

6) Express $\sqrt{98a^6}$ in simplest radical form.

7) Express $\sqrt{50x^5}$ in simplest radical form.

8) Simplify: $\sqrt{125x^4yz^5}$

9) Express $\sqrt[3]{27x^4y^9}$ in simplest radical form.

10) Express $2\sqrt[4]{48a^{12}b^7}$ in simplest radical form.

8.3 Adding Radicals

Learning Objectives: In this section, you will:

- Add and Subtract radical expressions with and without variables

Adding radicals as like terms involves combining expressions with the same radicand (the number inside the radical symbol) and the same index (the root).

The key is to combine the coefficients while preserving the radicand and the index.

1. Simplify your radical expressions.
2. If the radicands and indices match, you can treat them as like terms and perform the addition or subtraction accordingly.
   (If the radicands or indexes do not match you cannot combine your radical expressions.)

Example A) Add: $\sqrt{3} + \sqrt{3}$

- Explanation: When you add like terms, you add the coefficients (numbers outside the radicals) while keeping the radicand and the index the same.
- Solution: $2\sqrt{3}$

Example B) Add: $2\sqrt{5} + 4\sqrt{5}$

- Explanation: Here, you add the coefficients (2 and 3) while keeping the radicand (√5) and index (2) the same.
- Solution: $6\sqrt{5}$

Example C) Subtract: $4\sqrt[3]{2} - \sqrt[3]{2}$

- Explanation: Subtract the coefficients (4 - 1) while keeping the radicand ($\sqrt[3]{2}$) and index (3) the same.
- Solution: $3\sqrt[3]{2}$

Example D) Simplify: $4\sqrt{5} - 3\sqrt{2} + \sqrt{5} + 2\sqrt{2}$

- Combine like radicals. $4\sqrt{5} + 1\sqrt{5} = 5\sqrt{5}$ and $-3\sqrt{2} + 2\sqrt{2} = -1\sqrt{2}$
- Solution: $5\sqrt{5} - \sqrt{2}$ (Cannot combine not like radicals.)

Example E) Add: $\sqrt{8} + \sqrt{32}$

- Explanation: Simplify each radical first: $\sqrt{8} = 2\sqrt{2}$ and $\sqrt{32} = 4\sqrt{2}$.
- Then, add the like terms, $2\sqrt{8} + 4\sqrt{2} = 6\sqrt{2}$
- Solution: $6\sqrt{2}$

Example F) Simplify: $\sqrt{a^2b} + \sqrt{ab}$

- Explanation: Here, you cannot simplify further since the radicands (a^2b and ab) are different.
- Solution: $\sqrt{a^2b} + \sqrt{ab}$

Example G) Simplify: $2a\sqrt[3]{b^2} - \sqrt[3]{a^3b^2}$

1) Explanation: Explanation: Simplify each radical first: you cannot simplify the first, $\sqrt[3]{a^3b^2} = a\sqrt[3]{b^2}$
2) $2a\sqrt[3]{b^2} - a\sqrt[3]{b^2}$
3) Subtract the coefficients: 2 – 1 = 1 while keeping the radicand $a\sqrt[3]{b^2}$ and index: 3 the same.
4) Solution: $a\sqrt[3]{b^2}$

Example H) Simplify: $8\sqrt{5} - \sqrt{45} - \sqrt{80}$

- Simplify the radicals: $8\sqrt{5} - \sqrt{3 \cdot 3 \cdot 5} - \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5} = 8\sqrt{5} - 3\sqrt{5} - 4\sqrt{5}$
- Combine like radicals.
- Solution: $\sqrt{5}$

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<u>Worksheet: 8.3 Adding Radicals</u>

Simplify and add/subtract:

1) $2\sqrt{3} + 3\sqrt{3}$
2) $\sqrt{3} + \sqrt{2}$
3) $4\sqrt{3a} - \sqrt{3a}$
4) $-\sqrt{20} + 3\sqrt{5}$
5) $4\sqrt[3]{16} - \sqrt[3]{54}$
6) $4\sqrt{3} - 5\sqrt{7} + 5\sqrt{3} + 2\sqrt{7}$
7) $-3\sqrt{18} + 3\sqrt{50}$
8) $2\sqrt[3]{b^5} - b\sqrt[3]{4b^2}$
9) $4\sqrt{50x} + 5\sqrt{27} - 3\sqrt{2x} - 2\sqrt{108}$
10) $\sqrt[3]{5} + 3\sqrt{5} - 8\sqrt[3]{5} + 2\sqrt{5}$
11) $\sqrt[3]{81x^3y} - 3y\sqrt[3]{32x^2} + x\sqrt[3]{24y}$

8.4 Multiplying Radicals

Learning Objectives: In this section, you will:

- Multiply radical expressions with and without variables

"Product Rule" for radicals: when you multiply two radicals with the same index (root), you can simplify the expression by multiplying the contents (the numbers or variables) inside the radicals together under a single radical. Here's the rule in mathematical notation:

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b} = \sqrt{ab}$$

In words, this rule tells us that the product of two radicals ($\sqrt{a}$ and $\sqrt{b}$) is equal to the square root of the product of their contents $\sqrt{ab}$.

If the radicals have coefficients, multiply coefficients separately and multiply the radicals.

$$c\sqrt{a} \cdot d\sqrt{b} = c \cdot d\sqrt{a \cdot b} = cd\sqrt{ab}$$

Here's a breakdown of the steps for multiplying radicals:

Example A) Multiply Radicals with Numbers: $\sqrt{5} \cdot \sqrt{10}$

1. Multiply the numbers inside the radicals: $\sqrt{5} \cdot \sqrt{10} = \sqrt{50}$
2. Simplify the result: $\sqrt{2 \cdot 5 \cdot 5} = 5\sqrt{2}$ (you cannot simplify further)

Example B) Multiply Radicals with Coefficients: $2\sqrt{3} \cdot 3\sqrt{5}$

1. If there are any coefficients or constants outside the radicals, you should multiply them together. $2\sqrt{3} \cdot 3\sqrt{5}$, you can multiply 2 and 3 to get 6.
2. Then, you multiply the contents inside the radicals together. In our example, $\sqrt{3} \cdot \sqrt{5}$becomes $\sqrt{3 \cdot 5} = \sqrt{15}$.
3. If further simplification is possible (e.g., simplifying the radical), you should do that. In this case, $\sqrt{15}$ cannot be simplified further.
4. So, $2\sqrt{3} \cdot 3\sqrt{5}$ simplifies to $6\sqrt{15}$.

Example C) Multiply Radicals with Variables: $\sqrt{a} \cdot \sqrt{b^3}$

1. Multiply the variables inside the radicals: $\sqrt{a} \cdot \sqrt{b^3} = \sqrt{ab^3}$Take the square root of the result: $\sqrt{ab^3}$
2. Simplify: $\sqrt{ab^3} = \sqrt{a \cdot b \cdot b \cdot b} = b\sqrt{ab}$

Example D) Multiply Radicals with Numbers and Variables: $\sqrt{3x} \cdot \sqrt{6y}$

1. Multiply the numbers and variables inside the radicals: $\sqrt{3x \cdot 6y} = \sqrt{18xy}$
2. Take the square root of the result: $\sqrt{18xy}$
3. Simplify the radical: $\sqrt{2 \cdot 3 \cdot 3xy} = 3\sqrt{2xy}$

Example E) Multiply Radicals with Variables with Exponents: $\sqrt{a^3} \cdot \sqrt{a^5}$

1. Multiply the variables with exponents inside the radicals: $\sqrt{a^3} \cdot \sqrt{a^5} = \sqrt{a^3 a^5}\ (add\ exponents) = \sqrt{a^{3+5}} = \sqrt{a^8}$
2. Take the square root of the result, simplify: $\sqrt{a^8} = a^4$

Example F) Distribute Radicals (multiply 1 term by 2 terms): $2\sqrt{2}(5\sqrt{10} + \sqrt{14})$

1. Distribute to both terms: $2\sqrt{2}(5\sqrt{10} + \sqrt{14}) = 10\sqrt{20} + 2\sqrt{28}$
2. Simplify both radicals: $10\sqrt{20} + 2\sqrt{28} = 10\sqrt{4 \cdot 5} + 2\sqrt{4 \cdot 7} = 20\sqrt{5} + 4\sqrt{7}$

Example G) Distribute Radicals (multiply 2 terms by 2 terms): $(2\sqrt{2} + 4)(\sqrt{6} + 5)$

1. Distribute using FOIL: $(2\sqrt{2} + 4)(\sqrt{6} + 5) = 2\sqrt{12} + 10\sqrt{2} + 4\sqrt{6} + 20$
2. Simplify: $2\sqrt{12} + 10\sqrt{2} + 4\sqrt{6} + 20 = 2\sqrt{4 \cdot 3} + 10\sqrt{2} + 4\sqrt{6} + 20 = 4\sqrt{3} + 10\sqrt{2} + 4\sqrt{6} + 20$

Example H) Distribute: $(6 + 7\sqrt{7})^2$

1. Distribute using FOIL: $(2\sqrt{2} + 4)(\sqrt{6} + 5) = 2\sqrt{12} + 10\sqrt{2} + 4\sqrt{6} + 20$
2. Simplify: $2\sqrt{12} + 10\sqrt{2} + 4\sqrt{6} + 20 = 2\sqrt{4 \cdot 3} + 10\sqrt{2} + 4\sqrt{6} + 20 = 4\sqrt{3} + 10\sqrt{2} + 4\sqrt{6} + 20$

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<u>Worksheet: 8.4 Multiplying Radicals</u>

Multiply:

1) $\sqrt{2} \cdot \sqrt{2}$
2) $2\sqrt{3} \cdot 5\sqrt{6}$
3) $\sqrt{3x} \cdot 5\sqrt{x}$
4) $\sqrt{3ab} \cdot 5\sqrt{6a}$
5) $-3\sqrt[4]{8} \cdot 7\sqrt[4]{10}$
6) $-5\sqrt{3} \cdot 5\sqrt{3}$
7) $\sqrt{40x^2} \cdot \sqrt{4x^5}$
8) $\sqrt{11}(10 - \sqrt{11})$
9) $(2 + \sqrt{5})(-10 + \sqrt{5})$
10) $(6 - \sqrt{5})(6 + \sqrt{5})$
11) $(x + \sqrt{y})(\mathrm{x} - \sqrt{y})$
12) $(\sqrt{5} + 14\sqrt{7})(\sqrt{5} + \sqrt{7})$
13) $(\sqrt{5} - \sqrt{7})^2$

8.5 Dividing Radicals, Rationalizing Denominator

Learning Objectives: In this section, you will:

- Divide radical expressions with and without variables
- Divide by monomial (one term)
- Divide by binomial (two terms)
- Rationalize denominator

Dividing Radical Expressions (Quotient Rule):
"Quotient Rule" for radicals: when you divide two radicals with the same index (root), you can simplify the expression by dividing/simplifying the contents (the numbers or variables) inside the radicals together under a single radical. Here's the rule in mathematical notation:

$$\boxed{\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}}$$

In words, this rule tells us that the quotient of two radicals: $\sqrt{a}$ and $\sqrt{b}$ is equal to the square root of the quotient of their contents $\sqrt{\frac{a}{b}}$.

If the radicals have coefficients:

- divide/simplify coefficients separately and
- divide/simplify the radicals.

Example A) Divide Numbers: $\frac{\sqrt{16}}{\sqrt{4}}$

1) Divide the numbers inside the radicals: $\frac{\sqrt{16}}{\sqrt{4}} = \sqrt{\frac{16}{4}} = \sqrt{4}$
2) Take the square root of the result: $\sqrt{4} = 2$

Example B) Divide Variables with Exponents: $\frac{\sqrt{x^7}}{\sqrt{x^2}}$

1) Divide the variables inside the radicals: $\frac{\sqrt{x^7}}{\sqrt{x^2}} = \sqrt{\frac{x^7}{x^2}}$
2) Simplify (subtract exponents): $\sqrt{\frac{x^7}{x^2}} = \sqrt{x^{7-2}} = \sqrt{x^5}$
3) Simplify the radical: $\sqrt{x^5} = x^2\sqrt{x}$

Example C) Divide Numbers and Variables: $\frac{3\sqrt{20ab}}{6\sqrt{5a}}$

1) Simplify coefficients: $\frac{3}{6} = \frac{1}{2}$
2) Divide the numbers and variables inside the radicals: $\sqrt{\frac{20ab}{5a}} = \sqrt{4b}$
3) Take the square root of the result (do not forget the coefficients) and simplify:
$$\frac{1\sqrt{4b}}{2} = \frac{2\sqrt{b}}{2} = \sqrt{b}$$

Example D) Simplify the radical expression (multiple terms in numerator): $\frac{6-\sqrt{27}}{3}$

1) Simplify numerator: $\frac{6-\sqrt{9\cdot 3}}{3} = \frac{6-3\sqrt{3}}{3}$
2) Factor numerator $\frac{6-3\sqrt{3}}{3} = \frac{3(2-\sqrt{3})}{3}$
3) Simplify by dividing numerator and denominator by GCF: $\frac{3(2-\sqrt{3})}{3} = (2-\sqrt{3})$

Rationalizing Denominator
<u>Dividing Radical Expressions by Monomial</u>
Rationalizing the denominator is a mathematical process used to simplify or eliminate radicals (square roots, cube roots, etc.) from the denominator of a fraction.
To rationalize the denominator, you typically multiply both the numerator and denominator of a fraction by a carefully chosen expression that eliminates the radical from the denominator.

Example E) Rationalize: $\sqrt{\frac{3}{2}}$

1) **Rationalized Form:** multiply both the numerator and denominator by $\sqrt{2}$ (the conjugate of the denominator $\sqrt{2}$) to rationalize the denominator.

$$\sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{\sqrt{4}}$$

2) Simplify: $\frac{\sqrt{6}}{\sqrt{4}} = \frac{\sqrt{6}}{2}$

   The simplified form is $\frac{\sqrt{6}}{2}$. (Realize: there is not any radical in the denominator.)

Example F) Divide Numbers with Different Radicals: $\frac{\sqrt{9}}{\sqrt{27}}$

1) Divide the numbers inside the radicals: $\frac{9}{27} = \frac{1}{3}$
2) Take the square root of the result: $\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{\sqrt{4}} = \frac{1}{\sqrt{3}}$
3) Rationalize the denominator by multiplying both the numerator and denominator by $\sqrt{3}$:

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{9}}$$

4) Simplify: $\frac{\sqrt{3}}{3}$

Rationalizing Denominator
<u>Dividing Radical Expressions by Binomial (Use Conjugate)</u>
The main goal is to eliminate the radical from the denominator. This often involves using the difference of squares formula, which states that $(a + b)(a - b) = a^2 - b^2$.
To rationalize the radical, you multiply both the numerator and the denominator of the expression by the conjugate of the denominator. For example, if you have the expression a-√b, you would multiply it by the conjugate, which is a+√b. (Conjugate uses the opposite signs between the terms.)

Example G) Rationalize the denominator (binomial expression in denominator): $\frac{2+\sqrt{3}}{2-\sqrt{2}}$

1. **1)** Multiply by the conjugate of the denominator: Multiply both the numerator and denominator by $(2+\sqrt{2})$, which is the conjugate of $(2+\sqrt{2})$:
$\frac{2+\sqrt{3}}{2-\sqrt{2}} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}}$
2. **2)** Simplify the expression:
$$\frac{(2+\sqrt{3})}{(2-\sqrt{2})} \cdot \frac{(2+\sqrt{2})}{(2+\sqrt{2})} = \frac{4+2\sqrt{2}+2\sqrt{3}+\sqrt{6}}{4+2\sqrt{2}-2\sqrt{2}-\sqrt{4}} = \frac{4+2\sqrt{2}+2\sqrt{3}+\sqrt{6}}{4-\sqrt{4}}$$
$$= \frac{4+2\sqrt{2}+2\sqrt{3}+\sqrt{6}}{4-2} = \frac{4+2\sqrt{2}+2\sqrt{3}+\sqrt{6}}{2}$$
Realize: there is not any radical in the denominator.

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<u>Worksheet:</u> <u>8.5 Dividing Radicals, Rationalizing Denominator</u>
<u>Divide and rationalize the denominator:</u>

1) $\sqrt{\frac{36}{100}}$

2) $\frac{3\sqrt{24}}{15\sqrt{6}}$

3) $\frac{6\sqrt{x^4y}}{\sqrt{4x^2y^3}}$

4) $\frac{\sqrt[3]{12ab^3c^8}}{\sqrt[3]{2a^4c}}$

5) $\frac{12+8\sqrt{6}}{4}$

6) $\frac{10+5\sqrt{7}}{5}$

7) $\sqrt{\frac{7}{8}}$

8) $\sqrt{\frac{y^6}{x^7}}$

9) $\sqrt[3]{\frac{2}{9x}}$

10) $\frac{3}{\sqrt{2}-5}$

11) $\frac{3}{\sqrt{3}+\sqrt{10}}$

12) $\frac{1+\sqrt{5}}{3-\sqrt{5}}$

13) $\frac{3+\sqrt{6}}{5-\sqrt{24}}$

14) $\frac{x+\sqrt{y}}{x-\sqrt{y}}$

9.1 Quadratics - Solving with Radicals

Learning Objectives: In this section, you will:
- Solve equations with radicals.
- Check for extraneous solutions.

An equation that contains a variable expression in a radical is a radical equation.

$\sqrt[n]{ax+b} = c$; n is called Index or Root.

Solving equations requires isolation of the variable. Equations that contain a variable inside of a radical require algebraic manipulation of the equation so that the variable "comes out" from underneath the radical(s). This can be accomplished by raising both sides of the equation to the "nth" power, where n is the "index" or "root" of the radical. When the index is a 2 (i.e. a square root), we call this method "squaring both sides." Sometimes the equation may contain more than one radical expression, and it is possible that the method may need to be used more than once to solve it. When the index is an even number (n = 2, 4, etc.) this method can introduce extraneous solutions, so it is necessary to verify that any answers obtained actually work. Plug the answer(s) back into the original equation to see if the resulting values satisfy the equation. It is also good practice to check the solutions when there is an odd index to identify any algebra mistakes.

General Solution Steps:

Step 1. Isolate the Radical(s) and identify the index (n).
Step 2. Raise both sides of the equation to the "nth" power.
Step 3. Use algebraic techniques (i.e., factoring, combining like terms,) to isolate the variable. Repeat Steps 1 and 2 if necessary.
Step 4. Check answers. Eliminate any extraneous solutions from the final answer.

Example A) Solve $\sqrt{5-x} - 3 = 0$

Step1: Isolate the Radical $\quad \sqrt{5-x} = 3$

Step 2: Square both Sides $\quad (\sqrt{5-x})^2 = 3^2$

Step 3: Solve for x $\quad 5 - x = 9$

$-x = 4$

$x = -4$

Step 4: Check Answers $\quad \sqrt{5-(-4)} - 3 = 0$

$\sqrt{9} - 3 = 0$, then 3=3, so x=-4 is the solution.

Example B) Solve: $\sqrt{7x+2} = 4$

$\sqrt{7x+2}^{2} = 4^2$	Square both sides
$7x + 2 = 16$	Solve for x.
$7x = 14$	Divide both sides by 7.
$x = 2$	Check answer.
$\sqrt{7(2)+2} = 4$	
$\sqrt{16} = 4$	So, x=2 is the solution.

Example C) Solve $\sqrt[3]{x-1} = -4$

$\sqrt[3]{x-1} = -4$	Odd index, we don't have to check answer.
$\sqrt[3]{x-1}^{3} = (-4)^3$	Cube both sides, simplify exponents.
$x - 1 = -64$	Solve for x.
$x = -63$	This is the solution.

Example D) Solve for $\sqrt{x^2-2} - \sqrt{x+4} = 0$

$\sqrt{x^2-2} - \sqrt{x+4} = 0$	Isolate the radicals so that they are on opposite sides of the equal sign.
$\sqrt{x^2-2} = \sqrt{x+4}$	Square both Sides
$\sqrt{(x^2-2)^2} = \sqrt{(x+4)^2}$	
$x^2 - 2 = x + 4$	Solve for *x*.
$x^2 - x - 6 = 0$	Factor
$(x+2)(x-3) = 0$	
$x = -2, x = 3$	Check answers.

$\sqrt{(-2)^2-2} - \sqrt{(-2)+4} = 0$ *if x=-2*

$\sqrt{4-2} - \sqrt{(-2)+4} = 0$

$\sqrt{4} - \sqrt{2} = 0$

So, x=-2 is a solution

$\sqrt{(3)^2-2} - \sqrt{(3)+4} = 0$ *if x=3*

$\sqrt{9-2} - \sqrt{(3)+4} = 0$

$\sqrt{7} - \sqrt{7} = 0$

So, x=3 is also solution

Example E) Solve for $\sqrt{x-6}-\sqrt{x+9}+3$

$\sqrt{x-6}-\sqrt{x+9}+3=0$ Isolate the radicals so that they are on opposite sides of the equal sign.

$\sqrt{x-6}+3=\sqrt{x+9}$ Square both sides.

$(\sqrt{x-6}+3)^2=(\sqrt{x+9})^2$ Recall: $(a-b)^2=a^2-2ab+b^2$

The first term will need to be distributed using FOIL.

$$FOIL: \left(\sqrt{x-6}+3\right)\left(\sqrt{x-6}+3\right)=\sqrt{x-6}^{\,2}+3\sqrt{x-6}+3\sqrt{x-6}+9$$

$x-6+6\sqrt{x-6}+9=x+9$ Radical still remains, repeat step1 and step 2

$\sqrt{x-6}=1$

$(\sqrt{x-6)}^2=1^2$ Square both sides again.

$x-6=1$ Solve for x.

$x=7$ Check the answer, and this is the solution.

Example F) Solve for x: $x+\sqrt{4x+1}=5$

$x+\sqrt{4x+1}=5$ Isolate radical by subtracting x from both sides.

$\sqrt{4x+1}=5-x$ Square both sides.

$\sqrt{(4x+1)^2}=(5-x)^2$ Use Foil method $(a-b)^2=a^2-2ab+b^2$

$4x+1=25-10x+x^2$ Re-order terms.

$x^2-14x+24=0$ Factor.

$(x-12)(x-2)=0$ Set each factor equal to zero.

$x-12=0\ or\ x-2=0$ Solve each equation.

$x=12, x=2$ Check answers in original problem.

$12+\sqrt{4(12)+1}=5$ if *x=12,*
$12+\sqrt{48+1}=5$
$12+7=5$
19=5 False, extraneous root so *x=12 is not an answer.*
$2+\sqrt{4(2)+1}=5$ if *x=2*
$2+\sqrt{8+1}=5$
$2+3=5$
5=5 True. x=*2* is the solution.

Worksheet 9.1: Quadratics - Solving with Radicals

1) $\sqrt{x - 3} = 5$

2) $\sqrt{2x + 3} - 3 = 0$

3) $\sqrt{6x - 5} - x = 0$

4) $3 + x = \sqrt{6x + 13}$

5) $\sqrt{3 - 3x} - 1 = 2x$

6) $\sqrt{4x + 5} - \sqrt{x + 4} = 2$

7) $\sqrt{2x + 2} = 3 + \sqrt{2x - 1}$

8) $x + 4 = \sqrt{x + 10}$

9) $\sqrt{7 - 4x} = 2\sqrt{3}$

10) $\sqrt[3]{2x - 2} = 2$

11) $\sqrt[3]{x - 1} = \sqrt[3]{2x + 1}$

9.2 Solving with Exponents using the Square Root Property

Learning Objectives: In this section, you will:

- Solve quadratic equations of the form $x^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x-h)^2=k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

Example A) **Solve $x^2 = 9$ using factoring**:

$x^2 - 9 = 0$		Set to 0.
$(x-3)(x+3) = 0$		Factor.
$x-3 = 0$ or	$x+3 = 0$	Set each factor to zero.
$x = 3$ or	$x = -3$	Solve for x.
		You have two solutions: $x = 3, -3$

Factoring can be used if 'k' is a perfect square. In case of any 'k' (perfect square or not) we can use the Square Root Property.

> Use the **Square Root Property**:
>
> If $x^2 = k$, then x is a positive or negative square root of k. $x = \pm\sqrt{k}$

Given a quadratic equation with an x^2 term but no 'x' term, use the square root property to solve it.

1. Isolate the x^2 term on one side of the equal sign.
2. Take the square root of both sides of the equation, putting a ± sign before the expression on the side opposite the squared term.
3. Simplify the numbers on the side with the ± sign.

Example B) Solve a Simple Quadratic Equation Using the Square Root Property: $x^2 = 9$

Solution:

$x^2 = 9$	Take the square root of both sides. Remember to use a ± sign before the radical symbol.
$\sqrt{\boldsymbol{x^2}} = \pm\sqrt{\mathbf{9}}$	Then simplify the radical.
$\boldsymbol{x = \pm 3}$	So, the solutions are $\boldsymbol{x = 3 \ or -3}$

Realize that we got the same solutions as with factorization.

Example C) Solve a Simple Quadratic Equation Using the Square Root Property: $x^2 = 8$

Solution:

$x^2 = 8$	Take the square root of both sides. Remember to use a ± sign before the radical symbol.
$\sqrt{\boldsymbol{x^2}} = \pm\sqrt{\mathbf{8}}$	Then simplify the radical.
$\boldsymbol{x = \pm 2\sqrt{2}}$	So, the solutions are $\boldsymbol{x = 2\sqrt{2} \ or -2\sqrt{2}}$

Note: Realize that this problem cannot be solved with factorization over rational numbers.

Example D) Solve the Quadratic Equation Using the Square Root Property: $(x-4)^2 = 36$

Solution:

$\boldsymbol{(x-4)^2 = 36}$	Take the square root of both sides. Remember to use a ± sign before the radical symbol.
$\boldsymbol{\sqrt{(x-4)^2} = \pm\sqrt{36}}$	Then simplify the radical.
$\boldsymbol{x-4 = \pm 6}$	Split and solve the first-degree equations.
$\boldsymbol{x-4=6}$ or $\boldsymbol{x-4=-6}$	Solve.
x = 6+4 or x = -6+4	So, the solutions are $\boldsymbol{x = 10\ or -2}$

Note: Realize that this problem could be also solved by distributing, setting to zero, and factoring, since the roots are real numbers.

Example E) Solving Using the Square Root Property: $4(a+6)^2 + 20 = 720$

Solution:

$4(a+6)^2 + 20 = 720$	Isolate the square term.
$\boldsymbol{4(a+6)^2 = 720 - 20}$	Subtract 20 from both sides.
$\boldsymbol{4(a+6)^2 = 700}$	
$\boldsymbol{\frac{4(a+6)^2}{4} = \frac{700}{4}}$	Divide both sides by 4
$\boldsymbol{(a+6)^2 = 175}$	Take the square root of both sides. Remember to use a ± sign before the radical symbol.
$\boldsymbol{\sqrt{(a+6)^2} = \pm\sqrt{175}}$	Take the square root of both sides. Remember to use a ± sign before the radical symbol.
$\boldsymbol{a + 6 = \pm 5\sqrt{7}}$	Then simplify the radical. $\boldsymbol{\pm\sqrt{175} = \pm 5\sqrt{7}}$
$\boldsymbol{a = -6 \pm 5\sqrt{7}}$	Solve the first-degree equations (since you cannot combine like terms, you can just subtract 6 from both sides and you do not have to split your solutions).

$\boldsymbol{a = -6 + 5\sqrt{7}\ or\ a = -6 - 5\sqrt{7}}$ Another way to write your solutions.

Realize that your answers are irrational numbers, so this problem cannot be solved with factorization over rational numbers.

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Worksheet: 9.2 Solving with Exponents using the Square Root Property

Solve the equations using the Square Root Property:

1) $(x-4)^2 = 36$
2) $(a-2)^2 = 75$
3) $3(x-2)^2 = 48$
4) $-2(x+5)^2 = -16$
5) $(7h+4)^2 = 9$
6) $4(k-7)^2 - 10 = 350$
7) $(2a+4)^2 = 49$
8) $(-2x+5)^3 = 27$
9) $3(3x-7)^2 - 6 = 21$
10) $2m^3 - 2 = -18$

9.3 Completing the Square

Learning Objectives: In this section, you will:

- Solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square

All equations cannot be factored, only that ones that have rational roots.

Consider the following equation: $x^2 - 2x - 7 = 0$. The equation cannot be factored, however there are two irrational solutions to this equation, $1 + 2\sqrt{2}$ and $1 - 2\sqrt{2}$. To find these two solutions we will use a method known as completing the square. When completing the square, we will change the quadratic into a perfect square which can easily be solved with the square root property learned in 9.3.

Example A) Solve the quadratic equation by completing the square: $x^2 - 2x - 7 = 0$

1. Start with the quadratic equation: $x^2 - 2x - 7 = 0$
2. Move the constant term (in this case, 7) to the other side of the equation by adding it to both sides: $x^2 - 2x = 7$
3. To complete the square, we need to make the coefficient of the x^2 term (in this case, 1) equal to 1. Since it's already 1, we move to the next step.
4. Focus on the coefficient of the x term (in this case, -2). Divide it by 2 and square the result: $(-2 / 2)^2 = (-1)^2 = 1$
5. Add this value to both sides of the equation: $x^2 - 2x + 1 = 7 + 1$
6. Now, you have a perfect square trinomial on the left side: $(x - 1)^2 = 8$
7. Take the square root of both sides to solve for x: $x - 1 = \pm 2\sqrt{2}$
8. Solve for x: $x = 1 \pm 2\sqrt{2}$

So, the solution to the equation $x^2 - 2x - 7 = 0$ is $x = 1 \pm 2\sqrt{2}$.

Steps

Now we can **solve** a Quadratic Equation ($ax^2 + bx + c = 0$) in 5 steps by completing the square:

- **Step 1** Divide all terms by **a** (the coefficient of $\mathbf{x^2}$).
- **Step 2** Move the number term (**c/a**) to the right side of the equation.
- **Step 3** Complete the square on the left side of the equation and balance this by adding the same value to the right side of the equation.

We now have something that looks like $(x + p)^2 = q$, which can be solved this way:

- **Step 4** Take the square root on both sides of the equation. Do not forget +/- solutions.
- **Step 5** Subtract the number that remains on the left side of the equation to find **x**.

Example B) Solve: $5x^2 - 20x - 25 = 0$

Step 1 Divide all terms by 5:

$x^2 - 4x - 5 = 0$

Step 2 Move the number term to the right side of the equation:

$x^2 - 4x = 5$

Step 3 Complete the square on the left side of the equation and balance this by adding the same number to the right side of the equation:

$(b/2)^2 = (-4/2)^2 = (-2)^2 = 4$

$x^2 - 4x + 4 = 5 + 4$

$(x - 2)^2 = 9$

Step 4 Take the square root on both sides of the equation:

$x - 2 = \pm\sqrt{9} = \pm 3$

Step 5 Add 2 to both sides, so the solutions are:

$x = \pm 3 + 2 =$ **5 or -1**

(Note: since the solutions are rational numbers, the equation could be solved by factorization.)

Example C) Solve by completing the square: $2x^2 - 36x = -98$

Step 1 Divide all terms by 2:

$x^2 - 18x = -49$

Step 2 The number term is already on the right side of the equation:

$x^2 - 18x = -49$

Step 3 Complete the square on the left side of the equation and balance this by adding the same number to the right side of the equation:

$(b/2)^2 = (-18/2)^2 = (-9)^2 = 81$

$x^2 - 18x + 81 = -49 + 81$

$(x - 9)^2 = 32$

Step 4 Take the square root on both sides of the equation:

$x - 9 = \pm\sqrt{32}$ and simplify:

$x - 9 = \pm 4\sqrt{2}$

Step 5 Add 9 to both sides, so the solutions are:

$x = 9 \pm 4\sqrt{2}$

(Note: since the solutions are irrational numbers, the equation could not be solved by factorization.)

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Worksheet: 9.3 Completing the Square

1) Find the last term to make the trinomial into a perfect square: $x^2 + 10x +$ _____
   When this is factored, it becomes:

2) Find the last term to make the trinomial into a perfect square: $x^2 - 8x +$ ______
   When this is factored, it becomes:

3) Solve by completing the square: $x^2 + 12x = 13$

4) Solve by completing the square: $x^2 = -6x + 1$

5) Solve by completing the square: $2x^2 - 36x + 160 = 0$

6) Solve by completing the square: $4x^2 = -16x + 20$

7) Solve by completing the square: $x^2 = -8x + 15$

8) Solve by completing the square: $x^2 + 2\text{x} = -8x + 89$

9.4 Quadratic Formula

Learning Objectives: In this section, you will:
- Solve quadratic equations by using the quadratic formula.

The quadratic formula is a fundamental mathematical expression used to find the solutions or roots of a quadratic equation. A quadratic equation is a second-degree polynomial equation, typically written in the form: $\boldsymbol{ax^2+bx+c=0}$
In this equation: "a"," b" and "c" are coefficients, with "a "not equal to zero (a≠0).
"*x*" represents the variable we want to solve for.
The goal is to find the values of "*x*" that satisfy the equation, making it equal to zero.
The quadratic formula is expressed as:

$$\boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

In the formula, the $\pm$ symbol means that there are two possible solutions, one with the positive square root and one with the negative square root.

We can use the quadratic formula to solve any quadratic, this is shown in the following examples.

Example A) Solve $\boldsymbol{x^2+3x+2=0}$ using the Quadratic Formula.

$x=\frac{-3\pm\sqrt{3^2-4(1)(2)}}{2(1)}$	Substitute a=1, b=3, and c=2
$x=\frac{-3\pm\sqrt{9-8}}{2}$	Evaluate exponent and multiplication.
$x=\frac{-3\pm\sqrt{1}}{2}$	Evaluate root.
$x=\frac{-3\pm1}{2}$	Evaluate ± to get two answers.
$x=\frac{-2}{2}\ or\ x=\frac{-4}{2}$	Simplify fractions.
$x=-1\ \ or\ x=-2$	Our Solution.

Example B) Use the Quadratic Formula to solve for *x*: $\boldsymbol{3x^2-7x+2=0}$

$x=\frac{-(-7)\pm\sqrt{7^2-4(3)(2)}}{2(3)}$	Substitute a=3, b=-7, and c=2
$x=\frac{7\pm\sqrt{49-24}}{6}$	Evaluate exponent and multiplication.
$x=\frac{7\pm\sqrt{25}}{6}$	Evaluate root.
$x=\frac{7\pm5}{6}$	Evaluate ± to get two answers.
$x=\frac{12}{6}\ or\ x=\frac{2}{6}$	Simplify fractions.
$x=2\ \ or\ x=\frac{1}{3}$	Our Solution.

Example C) Solve $\mathbf{2x^2 - 5x - 3 = 0}$ using the Quadratic Formula.

$x = \frac{-(-5)\pm\sqrt{(-5)^2-4(2)(-3)}}{2(2)}$	Substitute a=2, b=-5, and c=-3
$x = \frac{5\pm\sqrt{25+24}}{4}$	Evaluate exponent and multiplication.
$x = \frac{5\pm\sqrt{49}}{4}$	Evaluate root.
$x = \frac{5\pm 7}{4}$	Evaluate ± to get two answers.
$x = \frac{12}{4}\ or\ x = \frac{-2}{4}$	Simplify fractions.
$x = 3\ \ or\ x = \frac{-1}{2}$	Our Solution.

Example D) Solve $\mathbf{3x^2 - 7 = 0}$ using the Quadratic Formula.

$x = \frac{-(0)\pm\sqrt{(0)^2-4(3)(-7)}}{2(3)}$	Substitute a=3, b=0 (missing term), and c=-7.
$x = \frac{0\pm\sqrt{0+84}}{6}$	Evaluate exponent and multiplication.
$x = \frac{\pm\sqrt{84}}{6}$	Simplify root.
$x = \frac{\pm 2\sqrt{21}}{6}$	Evaluate ± to get two answers.
$x = \frac{2\sqrt{21}}{6}\ or\ x = \frac{-2\sqrt{21}}{6}$	Simplify fractions.
$x = \frac{\sqrt{21}}{3}\ or\ x = \frac{-\sqrt{21}}{3}$	Our Solution.

Example E) Solve $\mathbf{4x^2 - 12x = -9}$ using the Quadratic Formula.

$4x^2 - 12x = -9$	First set equation equal to zero
$4x^2 - 12x + 9 = 0$	Add 9 to both sides.
$x = \frac{-(-12)\pm\sqrt{(-12)^2-4(4)(9)}}{2(4)}$	Substitute a=4, b=-12 and c=9.
$x = \frac{12\pm\sqrt{144-144}}{8}$	Evaluate exponent and multiplication.
$x = \frac{12\pm\sqrt{0}}{8}$	Evaluate root.
$x = \frac{12\pm 0}{8}$	Evaluate ± to get two answers.
$x = \frac{12}{8}$	Simplify fractions.
$x = \frac{3}{2}$	Our Solution.

Example F) Solve $\boldsymbol{3x^2+7=-x}$ using the Quadratic Formula.

$3x^2+x+7=0$	First set equation equal to zero
$x=\frac{-(1)\pm\sqrt{(1)^2-4(3)(7)}}{2(3)}$	Substitute a=3, b=1, and c=7.
$x=\frac{-1\pm\sqrt{1-84}}{6}$	Evaluate exponent and multiplication.
$x=\frac{-1\pm\sqrt{-83}}{6}$	Simplify root.
$x=\frac{1\pm\sqrt{-83}}{6}$	$\sqrt{-83}$ doesn't represent a real number.

Since $\sqrt{-83}$ doesn't represent a real number, there is no real number solution.

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Worksheet 9.4: Quadratic Formula

1) $x^2-4x+4 = 0$

6) $2x^2-x-3= 0$

2) $3x^2+6x+3 = 0$

7) $x^2-2x-4 = 0$

3) $2x^2-7x+3= 0$

8) $5x^2=10x-5$

4) $x^2+5x+6 = 0$

9) $2x^2+3x+1=0$

5) $3x^2+2x=1$

10) $3x^2-4x+2 = 0$

Beginning Algebra:
Answer Key for the Worksheets

6.1 Factoring-Greatest Common Factor

1) $9 + 8b^2$
2) $x - 5$
3) $5(9x^2 - 5)$
4) $1 + 2n^2$
5) $7(8 - 5p)$
6) $10(5x - 8y)$
7) $7ab(1 - 5a)$
8) $9x^2y^2(3y^3 - 8x)$
9) $3a^2b(-1 + 2ab)$
10) $4x^3(2y^2 + 1)$
11) $-5x^2(1 + x + 3x^2)$
12) $8n^5(-4n^4 + 4n + 5)$
13) $10(2x^4 - 3x + 3)$
14) $3(7p^6 + 10p^2 + 9)$
15) $4(7m^4 + 10m^3 + 2)$
16) $2x(-5x^3 + 10x + 6)$

6.2 Factoring-Grouping

1) $(8r^2 - 5)(5r - 1)$
2) $(5x^2 - 8)(7x - 2)$
3) $(n^2 - 3)(3n - 2)$
4) $(2v^2 - 1)(7v + 5)$
5) $(3b^2 - 7)(5b + 7)$
6) $(6x^2 + 5)(x - 8)$
7) $(3x^2 + 2)(x + 5)$
8) $(7p^2 + 5)(4p + 3)$
9) $(7x^2 - 4)(5x - 4)$
10) $(7n^2 - 5)(n + 3)$

6.3 Factoring-Trinomials where *a=1*

1) $(p+9)(p+8)$
2) $(x-8)(x+9)$
3) $(n-8)(n-1)$
4) $(x-5)(x+6)$
5) $(x+1)(x-10)$
6) $(x+5)(x+8)$
7) $a(b+8)(b+4)$
8) $2(b-10)(b-7)$
9) $(w-8)(w+1)$
10) $4b(a-5)(a-2)$

6.4 Factoring Trinomials where *a≠ 1*

1) (7x-6) (x-6)
2) (b+2) (7b+1)
3) (5a+7) (a-4)
4) (2x-1) (x-2)
5) (x+7) (2x+5)
6) (2b-3) (b+1)
7) (5k+3) (k+2)
8) (3x-5) (x-4)
9) (3x+2y) (x+5y)
10) (5x-7y) (x+7y)
11) 3(2x+1) (x-7)
12) 3(7k+6) (k-5)
13) 2(7x-2) (x-45)
14) (6x+5) (x+4)
15) (4k-1) (k-4)
16) (x+2y) (4x+y)
17) (4m+3n) (m-3n)
18) (x+3y) (4x+y)
19) 2(2x+7y) (3x+5y)
20) 4(6x-y) (x-2y)

6.5 Factoring Special Products

1) (r+4) (r-4)
2) (p+2) (p-2)
3) (3k+5) (3k-5)
4) 3(x+3) (x-3)
5) 4(x+3) (x-3)
6) 2(3a-5b) (3a-5b)
7) $(a-1)^2$
8) $(x+3)^2$

9) $(x-3)^2$
10) $(5a-1)^2$
11) $(5a+3b)^2$
12) $(2a-5b)^2$
13) $2(2x-3y)^2$
14) $-(m-2)(m^2+2m+4)$
15) $(x-4)(x^2+4x+16)$
16) $(5a-4)(25a^2+20a+16)$
17) $(4x+3y)(16x^2-12xy+9y^2)$
18) $(a^2+9)(a+3)(a-3)$
19) $-(z^2+4)(z+2)(z-2)$
20) $(m^2+9b^2)(m+3b$

6.6 Factoring Strategies

1) $3(2a+5y)(4z-3h)$
2) $(5u-4v)(u-v)$
3) $2(-x+4y)(x^2+4xy+16y^2)$
4) $n(5n-3)(n+2)$
5) $2(3u-2)(9u\ 2+6u+4)$
6) $n(n-1)$
7) $(x-3y)(x-y)$
8) $(3x+5y)(3x-5y)$
9) $(m+2n)(m-2n)$
10) $4(3b\ 2+2x)(3c-2d)$
11) $2(4+3x)(16-12x+9x\ 2)$
12) $2x(x+5y)(x-2y)$
13) $n(n+2)(n+5)$
14) $(3x-4)(9x^2+12x+16)$
15) $x(5x+2)$
16) $3k(k-5)(k-4)$
17) $(m-4x)(n+3)$
18) $(4x-y)^2$
19) $3(3m+4n)(3m-4n)$
20) $3x(3x-5y)(x+4y)$
21) $2(m-2n)(m+5n)$

6.7 Solve by Factoring

1) 7, -2
2) 1, -4
3) -5,5
4) 2, -7
5) $-\frac{5}{7}, -3$
6) $-\frac{1}{5}, 2$
7) 4,0
8) 1,4
9) $\frac{3}{7}, -8$
10) $\frac{4}{7}, -3$
11) -4, -3
12) 8, -2
13) $\frac{8}{3}, -5$
14) $-\frac{3}{7}, -3$
15) -4, 1
16) -7,7
17) $-\frac{5}{2}, -8$
18) $\frac{4}{5}, -6$

7.1 Factoring-Greatest Common Factor

1) $-\frac{1}{5}$
2) $\frac{13}{5}$
3) $\frac{19}{24}$
4) none
5) $-\frac{1}{2}, -1$
6) $\frac{1}{5}, 0$
7) $\frac{2x^2}{3}$
8) $-\frac{2a^2}{3b^2}$
9) $-\frac{4y^4}{3x^3z^3}$

10) $\frac{x+3}{2(x-3)}$
11) $\frac{x+5}{2}$
12) $\frac{3}{\mathrm{x}}$
13) $\frac{4}{x+3}$
14) $\frac{a-b}{a+b}$
15) $\frac{2x-3}{x-1}$
16) $\frac{5v-8}{5v+2}$
17) $\frac{1}{9}$
18) $\frac{3x-4}{10}$
19) $\frac{7y^2}{40y+18}$
20) $\frac{7x-6}{(3x+4)(x+1)}$
21) $\frac{7x-4}{4}$

7.2 Multiply and Divide Rational Expressions

1) $\frac{9x}{5y^3}$
2) $\frac{16a^2b^2}{3}$
3) $\frac{9x^2}{5}$
4) $\frac{5}{24}$
5) $\frac{x+7}{16x^2}$
6) $\frac{7(x-3)}{5(x+1)}$
7) $\frac{54}{35}$
8) $\frac{15}{2(x+4)}$
9) $\frac{2y-3}{y+3}$
10) $\frac{2z}{2\mathrm{z}-1}$
11) $-\frac{x-2}{x-1}$
12) $\frac{3}{3\mathrm{a}+1}$
13) $(x+5)(x-2)$
14) $-\frac{1}{y-x}$
15) $\frac{x}{x+3}$
16) $\frac{8}{r+6}$
17) $\frac{2}{(x-6)^2}$
18) n
19) $\frac{y-1}{6}$
20) $2a(a-8)$
21) $\frac{9}{r-6}$

7.3 Least Common Denominator

1) $6x^2y^6$
2) $x(x+5)$
3) $(x-5)(x+5)$
4) $(x+1)(x+2)(x+3)$
5) $4(x-2)$
6) $(x+1)^2(x+2)$
7) $\frac{6a^4}{10a^3b^2}, \frac{\mathrm{b}}{10a^3b^2}$
8) $\frac{3\mathrm{x}+1}{(x-5)(x-6)}, \frac{7\mathrm{x}-35}{(x-5)(x-6)}$
9) $\frac{x^2+4x+4}{(\mathrm{x}-4)(\mathrm{x}+2)}, \frac{x^2-8x+16}{(\mathrm{x}-4)(\mathrm{x}+2)}$
10) $\frac{5}{\mathrm{y}(\mathrm{y}-4)}, \frac{3y-12}{\mathrm{y}(\mathrm{y}-4)}, \frac{-2y}{\mathrm{y}(\mathrm{y}-4)}$

11) $\frac{5x+10}{x^3(x+2)}, \frac{x^3-3x^2}{x^3(x+2)}$

12) $\frac{4x}{(x-3)(x+2)}, \frac{x^2-6x+9}{(x-3)(x+2)}$

13) $\frac{z^2-5z}{(z-5)^2(z+5)}, \frac{2z^2+10z}{(z-5)^2(z+5)}$

14) $\frac{y}{5x^3y^3z}, \frac{10x^4}{5x^3y^3z}$

15) $\frac{7x+2}{(x+1)(x+2)(x+3)}, \frac{x+1}{(x+1)(x+2)(x+3)}$

16) $\frac{2x^2+3x+1}{(x-4)(x+3)(x+1)}, \frac{x^2+4x+3}{(x-4)(x+3)(x+1)}$

17) $\frac{x^2+4x-45}{3x(x-5)}, \frac{6x^2}{3x(x-5)}$

18) $\frac{4y+28}{3y(y-2)(y+7)}, \frac{3y^3}{3y(y-2)(y+7)}$

19) $\frac{7x^2-28}{(x+2)^2(x-2)^2}, \frac{6x^2-24x+24}{(x+2)^2(x-2)^2}, \frac{8x^2+32x+32}{(x+2)^2(x-2)^2}$

20) $\frac{6x^2+3x-9}{3(2x+1)^2(x-1)}, \frac{14x^2+7x}{3(2x+1)^2(x-1)}$

21) $\frac{48x^2-48}{30x(x-1)(x+1)}, \frac{5x^2-10}{30x(x-1)(x+1)}, \frac{10x^2+10}{30x(x-1)(x+1)}$

7.4 Add and Subtract Rational Expressions

1) $\frac{5y+6x}{10xy}$

2) $\frac{8-7}{6x^2}$

3) $\frac{xy^2+2y-x^3}{x^2y^2}$

4) $\frac{11x+53}{(x+5)(x+1)(x+4)}$

5) $\frac{-4x}{(x+1)(x-1)}$

6) $\frac{y^2-y+4}{(y-3)^2(y+3)}$

7) $\frac{2z^2-7z-4}{(z+2)(z-7)(z-1)}$

8) $\frac{x+10}{x-2}$

9) $\frac{35y^2+3x}{20x^2y^3}$

10) $\frac{20y^2+y+9}{5y}$

11) $\frac{y^2-8y-30}{(y+5)(y-2)}$

12) $\frac{2-x}{x-7}$

13) $\frac{xy^2-x^2y-y+x-x^3}{x(y-x)}$

14) $\frac{10}{(x-4)(x+4)(x-1)}$

15) $\frac{14x+11}{2x^2}$

16) $\frac{1}{x-7}$

17) $\frac{1}{y-2}$

18) $\frac{4x+22}{(x+1)(x+2)(x+3)(x-4)}$

19) $\frac{2x^2-14x-4}{(x-2)^2(x+2)}$

20) $\frac{2}{x+3}$

21) $\frac{4x^2-27x-17}{(2x+1)(x-7)}$

7.5 Complex Fractions

1) $\frac{56}{149}$

2) $-2m$

3) $\frac{x^3+12}{12x^2}$

4) $\frac{1-y}{y}$

5) $\frac{-2x+3}{5x+25}$
6) $\frac{4n^2-3n}{n+1}$
7) $\frac{x+y}{x-y}$
8) $-\frac{3}{x}$
9) $\frac{ab^2-a^2b}{a^2+b^2}$
10) $\frac{-4x+28}{9x+12}$
11) $\frac{-2x-42}{x^2-x-12}$
12) $\frac{y^2+5y+15}{10-y^2}$
13) $-\frac{1}{x(x+h)}$
14) $\frac{-2x-h}{x^2(x+h)^2}$
15) $\frac{y-2}{y+1}$
16) $\frac{x^2+9x+17}{x^2-7x-13}$
17) $\frac{x-3}{2}$
18) $\frac{a+7b}{8}$
19) $\frac{1}{x-1}$
20) $\frac{7}{4x+3}$
21) $\frac{5y-12}{3y+10}$

7.7 Solve Rational Equations

1) $\frac{3}{5}$
2) -7
3) $-\frac{89}{10}$
4) $\frac{1}{6}$
5) -2
6) $2,3$
7) $4,-1$
8) none
9) -7
10) $\frac{-2}{17}$
11) $-\frac{51}{9}$
12) $0,-\frac{1}{4}$
13) 8
14) $\frac{26}{5}$
15) $\frac{5}{3}$
16) *none*
17) none
18) -43
19) $-\frac{5}{14}$
20) -8

8.1 Square Roots and 8.2 Higher Roots

1) 8
2) $5\sqrt{3}$
3) $24\sqrt{2}$
4) $3\sqrt[3]{2}$
5) $-80\sqrt{6}$
6) $7a^3\sqrt{2}$
7) $5x^2\sqrt{2x}$
8) $5x^2z^2\sqrt{5z}$
9) $\sqrt[3]{27x^4y^9}$
10) $4a^3b^2\sqrt[4]{3b}$

8.3 Adding Radicals

1) $5\sqrt{3}$
2) $\sqrt{3}+\sqrt{2}$
3) $3\sqrt{3a}$
4) $\sqrt{5}$
5) $5\sqrt[3]{2}$
6) $9\sqrt{3}-3\sqrt{7}$
7) $6\sqrt{2}$
8) 0
9) $17\sqrt{50x}+3\sqrt{3}$
10) $-7\sqrt[3]{5}+5\sqrt{5}$
11) $5x\sqrt[3]{3y}-6y\sqrt[3]{4x^2}$

8.4 Multiply Radicals

1) 2
2) $30\sqrt{2}$
3) $5x\sqrt{3}$
4) $15a\sqrt{2b}$
5) $-42\sqrt[4]{5}$
6) -75
7) $4x^3\sqrt{10\mathrm{x}}$
8) $-11+10\sqrt{11}$
9) $-15-8\sqrt{5}$
10) 31
11) x^2-y
12) $103+15\sqrt{35}$
13) $-2-2\sqrt{35}$

8.5 Divide and rationalize the denominator:

1) $\frac{3}{5}$
2) $\frac{2}{5}$
3) $\frac{6x}{2y}$
4) $\frac{bc^2\sqrt[3]{6}}{a}$
5) $3+2\sqrt{6}$
6) $2+\sqrt{7}$
7) $\frac{\sqrt{14}}{2}$
8) $\frac{y^3\sqrt{x}}{x^4}$
9) $\frac{\sqrt[3]{6x^2}}{3x}$
10) $\frac{-3(\sqrt{2}+5)}{23}$
11) $\frac{-3(\sqrt{3}-\sqrt{10})}{7}$
12) $\frac{-9-4\sqrt{5}}{2}$
13) $27+10\sqrt{6}$
14) $\frac{x^2+2x\sqrt{y}+y}{x^2-y}$

9.1 Quadratics - Solving with Radicals

1) 2
2) 1,5
3) 2, -2
4) $\frac{1}{4}$
5) 5
6) 3
7) -1
8) $-\frac{5}{4}$
9) 5
10) -2

9.2 Solving with Exponents using the Square Root Property

1) x = -2, 10
2) $a = 2 - 5\sqrt{3},\ 2 + 5\sqrt{3}$
3) x = -2, 6
4) $x = -5 - 2\sqrt{2}, -5 + 2\sqrt{2}$
5) $h = -\frac{1}{7}, -1$
6) $x = 7 - 3\sqrt{10},\ 7 + 3\sqrt{10}$
7) $x = -\frac{11}{2}, \frac{3}{2}$
8) a = 1
9) $x = \frac{4}{3}, \frac{10}{3}$
10) m = -2

9.3 Completing the Square

1) 25; $(x+5)^2$
2) 16; $(x-4)^2$
3) x = 1, -13
4) $x = -3 + \sqrt{10}, -3 - \sqrt{10}$
5) x = 10, 8
6) x = -5, 1
7) $x = -4 + \sqrt{31}, -4 - \sqrt{31}$
8) $x = 3 + 7\sqrt{2},\ 3 - 7\sqrt{2}$

9.4 Quadratics - Quadratic Formula

1) 2
2) -1
3) 3, $\frac{1}{2}$
4) -2, -3
5) -1, $\frac{1}{3}$
6) -1, $\frac{3}{2}$
7) $1+\sqrt{5}, 1 - \sqrt{5}$
8) 1
9) -1, $-\frac{1}{2}$
10) No real number solution

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